MECHANICS;

OR, THE

DOCTRINE

OF

MOTION.

COMPREHENDING,

I. The general LAWS of MOTION.

II. The DESCENT of BODIES perpendicularly, and down inclined PLANES; and alſo in CURVE SURFACES. The MOTION of PENDULUMS.

III. CENTERS of GRAVITY. The EQUILIBRIUM of BEAMS of TIMBER, and their FORCES and DIRECTIONS.

IV. The MECHANICAL POWERS.

V. The comparative STRENGTH of TIMBER, and its STRESS. The POWERS of ENGINES, their MOTION, and FRICTION.

VI. HYDROSTATICS and PNEUMATICS.

Da veniam ſcriptis, quorum non gloria nobis
cauſa, ſed utilitas officiumque, fuit.

OVID DE PONTO III.

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THE

PREFACE.

HAVING some years since written a large book of Mechanics, which is sold by Messr. Robinson, *and* Co. *in* Pater-noster-Row. *I have here given a short abstract of that book, as it falls properly into this course; and especially as that branch of science, is of such extensive use in the affairs of human life. I do not in the least, design this to interfere with the other book, being rather an introduction to it, as it explains several things in it more at large; particularly in the first section, as being of universal extent and use; and likewise in several other parts of the book, especially such as have been objected to by ignorant writers. I have also added several things not mentioned in the other book, which are more simple and easy, and more proper for learners. So that this short treatise may be looked upon as an introduction to the other book, and will doubtless facilitate the reading of it. As to the higher and more difficult matters, as few care to trouble their heads about them, I have said little of them here, being not so proper for an introduction. To mention one or two things; I had taken a great deal of pains to find out the true form of a bridge, that shall be the strongest, and of a ship that shall sail the fastest; both upon principles that I know to be as certain and demonstrative as the Elements of Euclid; both these you have in the other book. But, as we have no occasion in* England, *for* *the ſtrongeſt bridges, or the ſwifteſt ſhips; Mathematicians, for the future, may find ſomething elſe to do, than run into ſuch perplext and uſeleſs diſquiſitions. For indeed when any of theſe grand things are to be performed, they generally fall into the hands of ſuch people, as know little of the nature of them. They perhaps know how to lay down upon paper, the plan of a deſign, by rule and compaſs, and to do ſeveral problems in practical geometry; and not much more. As if this was an adequate qualification, for conducting ſuch magnificent works, as coſt many thouſand pounds to execute. So that theſe things, inſtead of being conſtructed by the rules of art; they are too often done by fancy, without any true rules.*

But in this there is no great wonder, conſidering how few people ſtudy this art; and among theſe that do, how few are competent judges. For even among thoſe that preſume to write about it, it is ſurprizing to ſee what miſtakes they daily run into. One denies the compoſition and reſolution of forces; another cannot be ſatisfied with ſome of the mechanical powers, not even ſo ſimple a thing as the wedge; all owing to the wrong notions they have imbibed. And in conſequence of this, will be either condemning the art itſelf, or criticiſing on other writers, whom they do not underſtand.

As to what I have written on this ſubject; I have all along given the demonſtrations of the ſeveral things I have handled; and I expect that *to be my teſt, as to the truth or falſhood thereof. And by this teſt I leave them to be tried by any judicious, honeſt reader; who is a lover of truth, and a promoter of ſcience.*

W. Emerſon.

MECHA-

MECHANICS.

DEFINITIONS.

1. *MECHANICS* is a ſcience, which treats of the forces, motions, velocities, and in general, of the actions of bodies upon one another. It teaches how to move any given weight with any given power; how to contrive engines to raiſe great weights, or to perform any kind of motion.

2. *Body* is the maſs or quantity of matter; an *elaſtic body* is that which yields to a ſtroke, and recovers its figure again. But if not, 'tis called an *unelaſtic body*.

3. *Denſity* is the proportion of the quantity of matter in any body, to the quantity of matter in another body of the ſame bigneſs.

4. *Force* is a power exerted on a body to move it. If it act inſtantaneouſly, 'tis called *Percuſſion*, or *impulſe*. If conſtantly, 'tis an *accelerative force*.

5. *Velocity* is a property of motion, by which a body paſſes over a certain ſpace in a certain time. And is greater or leſſer, as it paſſes over a greater or leſſer ſpace in a certain time as ſuppoſe a ſecond.

6. *Motion* is a continual and ſucceſſive change of place. If the body moves equally, 'tis called *equable* or *uniform motion*. If it increaſes or decreaſes, 'tis called *accelerated* or *retarded motion*. When it is compared with ſome body at reſt, 'tis called *abſo-*

lute motion. But when compared with others in motion, it is called *relative motion*.

7. *Direction* of motion is the courſe or way the body tends, or the line it moves in.

8. *Quantity* of motion, is the motion a body has, conſidered both in regard to its velocity and quantity of matter. This is alſo called the *Momentum* of a body.

9. *Vis inertiæ*, is the innate force of matter, by which it reſiſts any change, ſtriving to preſerve its preſent ſtate of reſt or motion.

10. *Gravity* is that force wherewith a body endeavours to fall downwards. It is called *abſolute gravity* in empty ſpace; and *relative gravity* when immerſed in a fluid.

11. *Specific gravity*, is the greater or leſſer weight of bodies of the ſame magnitude, or the proportion between their weights. This proceeds from the natural denſity of bodies.

12. *Center of gravity*, is a certain point of a body; upon which, the body when ſuſpended, will reſt in any poſition.

13. *Center of motion*, is a fixed point about which a body moves. And the *axis of motion* is a fixed line it moves about.

14. *Power* and *weight*, when oppoſed to one another, ſignify the body that moves another, and the other which is moved. The body which begins and communicates motion is the *power*; and that which receives the motion, is the *weight*.

15. *Equilibrium* is the balance of two or more forces, ſo as to remain at reſt.

16. *Machine* or *Engine*, is any inſtrument to move bodies, made of levers, wheels, pullies, &c.

17. *Mechanic powers*, are the ballance, lever, wheel, pulley, ſcrew and wedge.

18. *Streſs* is the effect any force has to break a beam, or any other body; and *ſtrength* is the reſiſtance

resistance it is able to make against any straining force.

19. *Friction* is the resistance which a machine suffers, by the parts rubbing against one another.

POSTULATA.

1. That a small part of the surface of the earth may be looked upon as a plane. For tho' the earth be round, yet such a small part of it as we have any occasion to consider, does not sensibly differ from a plane.

2. That heavy bodies descend in lines parallel to one another. For tho' they all tend to a point which is the center of the earth, yet that center is at such a distance that these lines differ insensibly from parallel lines.

3. The same body is of the same weight in all places on or near the earth's surface. For the difference is not sensible in the several places we can go to.

4. Tho' all matter is rough, and all engines imperfect; yet for the ease of calculation, we must suppose all planes perfectly even; all bodies perfectly smooth; and all bodies and machines to move without friction or resistance; all lines streight and inflexible, without weight or thickness; cords extremely pliable, and so on.

AXIOMS.

1. Every body endeavours to remain in its present state, whether it be at rest, or moving uniformly in a right line.

2. The alteration of motion by any external force is always proportional to that force, and in direction of the right line in which the force acts.

3. Action and re-action, between any two bodies, are equal and contrary.

4. The motion of any body is made up of the ſum of the motions of all the parts.

5. The weights of all bodies in the ſame place, are proportional to the quantities of matter they contain, without any regard to their figure.

6. The vis inertiæ of any body, is proportional to the quantity of matter.

7. Every body will deſcend to the loweſt place it can get to.

8. Whatever ſuſtains a heavy body, bears all the weight of it.

9. Two equal forces acting againſt one another in contrary directions; deſtroy one anothers effects. And unequal forces act only with the difference of them.

10. When a body is kept in equilibrio; the contrary forces in any line of direction are equal.

11. If a certain force generate any motion; an equal force acting in a contrary direction, will deſtroy as much motion in the ſame time.

12. If a body be acted on by any power in a given direction. It is all one in what point of that line of direction, the power is applied.

13. If a body is drawn by a rope, all the parts of the rope are equally ſtretched. And the force in any part acts in direction of that part. And it is the ſame thing whether the rope is drawn out at length, or goes over ſeveral pullies.

14. If ſeveral forces at one end of a lever, act againſt ſeveral forces at the other end; the lever acts and is acted on in direction of its length.

SECT.

SECT. I.

The General Laws of MOTION.

PROP. I.

THE quantities of matter in all bodies, are in the compound ratio of their magnitudes and densities.

For (Def. 3.) in bodies of the same magnitudes, the quantities of matter will be as the densities. Increase the magnitude in any ratio, and the quantity of matter is increased in the same ratio. Consequently the quantity of matter is in the compound ratio of the density and magnitude.

Cor. 1. *In two similar bodies, the quantities of matter are as the densities, and cubes of the diameters.* For the magnitudes of bodies are as the cubes of the diameters.

Cor. 2. *The quantities of matter are as the magnitudes and specific gravities.* For (by Def. 3. and 11.) the densities of bodies are as their specific gravities.

PROP. II.

The quantities of motion in all moving bodies, are in the compound ratio of the quantities of matter and the velocities.

For if the velocities be equal, the quantities of motion will manifestly be as the quantities of matter. Increase the velocity in any ratio, and the quantity of motion will be increased in the same ratio. Therefore it follows univerſally, that the quantities of motion are in the compound ratio of the velocities and quantities of matter.

Cor. *Hence if the body be the ſame, the motion is as the velocity. And if the velocity be the ſame, the motion is as the body or quantity of matter.*

PROP. III.

In all bodies moving uniformly, the ſpaces deſcribed, are in the compound ratio of the velocities and the times of their deſcription.

For in any moving body, the greater the velocity, the greater is the ſpace deſcribed; that is, the ſpace will be as the velocity. And in twice or thrice the time, &c. the ſpace will be twice or thrice as great; that is, the ſpace will be increaſed in proportion to the time. Therefore univerſally the ſpace is in the compound ratio of the velocity, and the time of deſcription.

Cor. 1. *The time of deſcribing any ſpace, is as the ſpace directly and velocity reciprocally; or as the ſpace divided by the velocity. And if the velocity be the ſame, the time is as the ſpace. And if the ſpace be the ſame, the time is reciprocally as the velocity.*

Cor. 2. *The velocity of a moving body, is as the ſpace directly, and time reciprocally; or as the ſpace divided by the time. And if the time be the ſame, the velocity is as the ſpace deſcribed. And if the ſpace be the ſame, the velocity is reciprocally as the time of deſcription.*

PROP. IV.

The motion generated by any momentary force, or by a single impulse, is as the force that generates it.

For if any force generates any quantity of motion; double the force will produce double the motion; and treble the force, treble the motion, and so on If a body striking another, gives it any motion, twice that body striking the same, with the same velocity, will give it twice the motion, and so the motion generated in the other will be as the force of percussion.

Cor. 1. *Hence the forces are in the compound ratio of the velocities and quantities of matter.*

For (Prop. II.) the motions are as the quantities of matter multiplied by the velocities.

Cor. 2. *The velocity generated, is as the force directly, and quantity of matter reciprocally. Therefore if the bodies are equal, the velocities are as the forces. And if the forces are equal, the velocities are reciprocally as the bodies.*

Cor. 3. *The quantity of matter, is as the force directly, and velocity reciprocally. And therefore if the velocities be equal, the bodies are as the forces. And if the forces be equal, the bodies are reciprocally as the velocities.*

PROP. V.

The quantity of motion generated, by a constant and uniform force, is in the compound ratio of the force and time of acting.

For the motion generated in any given time will be proportional to the force that generates it;

 and

and in twice that time, the motion will be double by the same force; and in thrice that time it will be treble; and so every part of time adds a new quantity of motion equal to the first; and therefore the whole motion, will be as the force, and the whole time of acting.

Cor. 1. *The motion lost in any time, is in the compound ratio of the force and time.*

Cor. 2. *The velocity generated (or destroyed) in any time, is as the force and time directly, and quantity of matter reciprocally. The same is true of the increase or decrease of velocity.*

For the motion, that is (Prop. II.) the body multiplied by the velocity, is as the force and time. And therefore the velocity is as the force and time directly, and the body reciprocally.

Cor. 3. *Hence if the force be as the quantity of matter, the velocity is as the time. Or if the force and quantity of matter be given, the velocity is as the time.*

And if the time and quantity of matter be given, the velocity is as the force.

And if the force and time be given, the velocity is reciprocally as the matter.

Cor. 4. *The time is as the quantity of matter and velocity directly, and the force reciprocally. Therefore,*

If the force and velocity be given, the time is as the quantity of matter.

If the quantity of matter and velocity be given, the time is reciprocally as the force.

Cor. 5. *The force is as the quantity of matter and velocity directly, and the time reciprocally. Whence,*

If the velocity is at the time, or if the velocity and time be given, then the force is as the quantity of matter.

And

And if the velocity and quantity of matter be given, the force is reciprocally as the time.

Cor. 6. *The quantity of matter is as the force and time directly, and the velocity reciprocally. Therefore, if the force and time be given, the quantity of matter is reciprocally as the velocity.*

Cor. 7. *Hence also if the body be given, the velocity is in the compound ratio of the force and time.*

And if the force be given, the time is in the compound ratio of the matter and velocity, or as the quantity of motion.

PROP. VI.

If a given body is urged by a constant and uniform force; the space described by the body from the beginning of the motion, will be as the force and square of the time.

Suppose the time divided into an infinite number of equal parts or moments. Then in each of these moments of time, the space described (Prop. III. Cor. 2.) will be as the velocity gained; that is, (by Cor. 7. Prop. V.) as the force and time from the beginning. And the sum of all the spaces, or the whole space described, will be as the force and the sum of all the moments of time from the beginning. Therefore put t = the whole time, and the whole space described will be as the force and sum of the times 1, 2, 3, 4, &c. to t. But the sum of the arithmetic progression $1 + 2 + 3 + 4$ ... to $t = \frac{t + 1}{2} \times t = \frac{1}{2} tt$, because t is infinite or consists of an infinite number of moments. Therefore the whole space described will be as the force and $\frac{1}{2} tt$; that is, (because $\frac{1}{2}$ is a given quantity), as the force and the square of the time of description.

Cor.

Cor. 1. *If a body is impelled by a conſtant and uniform force; the ſpace deſcribed from the beginning of the motion, is as the velocity gained, and the time of moving.*

For the ſpace is as the force and ſquare of the time, or as the force × time × time. But (Cor. 7. Prop. V.) the force × time is as the velocity; therefore the ſpace which is as the force × time × time, is as the velocity × time.

Cor. 2. *If a body urged by any conſtant and uniform force, deſcribes any ſpace; it will deſcribe twice that ſpace in the ſame time, by the velocity acquired.*

For the ſum of all the ſpaces deſcribed by that force, 1 + 2 + 3 &c. to t, was ſhewn to be $\frac{1}{2}tt$. But the ſum of all the ſpaces deſcribed by the laſt velocity, will be $t + t + t$ &c. to t terms, whoſe ſum is tt. But tt is double to $\frac{1}{2}tt$; that is, the ſpace deſcribed by the laſt velocity, is double the ſpace deſcribed by the accelerating force.

Cor. 3. *Univerſally in all bodies urged by any conſtant and uniform forces; the ſpace deſcribed is as the force and ſquare of the time directly, and the quantity of matter reciprocally.*

For (Cor 1.) the ſpace is as the time and velocity. But (Prop V. Cor 2.) the velocity is univerſally as the force and time directly, and quantity of matter reciprocally. Therefore the ſpace is as the ſquare of the time and the force directly, and matter reciprocally; whence,

Cor. 4. *The product of the force, and ſquare of the time, is as the product of the body and ſpace deſcribed.*

Cor. 5. *The product of the force and time, is as the product of the quantity of matter and velocity.*

For (Prop. V.) the product of the force and

time,

time, is as the motion; that is, as the body and velocity. Fig.

Cor. 6. *The product of the body, and square of the velocity, is as the product of the force and the space described.*

For (Cor. 5.) the product of the body and velocity, is as the force and time. Therefore, the body × velocity square, is as the force × time × velocity; but time × velocity is as the space (by Prop. III.); therefore body × velocity square is as the force × space.

SCHOLIUM.

If any quantity or quantities are given, they must be left out. And such quantities as are proportional to each other must be left out. For example, if the quantity of matter be always the same; then (Cor. 3.) the space described is as the force and square of the time. And if the matter be proportional to the force, as all bodies are in respect to their gravity; then (Cor. 6.) the space described is as the square of the velocity. Or if the space described be always proportional to the body; then (Cor. 6.) the force is as the square of the velocity. Again, if the body be given, then (Cor. 4.) the space is as the force and square of the time. And if both the quantity of matter and the force be given, the space described is as the square of the time. And so of others.

PROP. VII.

If ABCD *be a parallelogram; and if a body at* A, *be acted upon separately by two forces, in the directions* AB *and* AC, *which would cause the body to be carried thro' the spaces* AB, AC *in the same time. Then both forces acting at once, will cause the body to be carried thro' the diagonal* AD *of the parallelogram.*

Let the line AC be supposed to move parallel to itself;

Fig. 1. itſelf; whilſt the body at the ſame time moves from A, along the line AC or *bg*, and comes to *d* at the ſame inſtant, that AC comes to *bg*. Then ſince the lines AB, AC, are deſcribed in the ſame time; and A*b*, A*d*, are alſo deſcribed in the ſame time. Therefore, as the motions are uniform, it will be, A*b* : *bd* : : AB : BD; and therefore A*d*D is a ſtreight line, coinciding with the diagonal of the parallelogram.

Cor. 1. *The three forces in the directions* AB, AC, AD, *are reſpectively as the lines* AB, AC, AD.

Cor. 2. *Any ſingle force* AD *denoted by the diagonal of a parallelogram, is equivalent to two forces denoted by the ſides* AB, AC.

Cor. 3. *And therefore any ſingle force* AD *may be reſolved into two forces, an infinite number of ways, by drawing any two lines* AB, BD, *for their quantities and directions.*

SCHOLIUM.

This practice of finding two forces equivalent to one, or dividing one force into two; is called the *compoſition* and *reſolution* of forces.

PROP. VIII.

If three forces A, B, C, *keep one another in equilibrio; they will be proportional to three ſides of a triangle, drawn parallel to their ſeveral directions,* DI, CI, CD.

Produce AD to I, and BD to H, and compleat the parallelogram DICH; then (Prop. VII.) the force in direction DC, is equal to the forces DH, DI, in the directions DH, DI. Take away the force DC and putting the forces DH, DI equal thereto;

thereto; and the equilibrium will still remain. Fig.
Therefore (Ax. 10.) DI is equal to the force A op- 2.
posite to it; and DH or CI equal to its opposite force B. And as CD represents the force C, the three forces A, B, and C will be to one another as DI, CI, and CD.

Cor. 1. *Hence if three forces acting against one another, keep each other in equilibrio; these forces will be respectively as the three sides of a triangle drawn perpendicular to their lines of direction; or making any given angle with them, on the same side.*

For this triangle will be similar to a triangle whose sides are parallel to the lines of direction.

Cor. 2. *If three active forces* A, B, C *keep one another in equilibrio; they will be respectively as the sines of the angles, which their lines of direction pass through.*

For A, B, C are as DI, CI, CD; that is, as S.DCI, S.CDI, and S.DIC. But S.DCI = S.CDH = S.CDB. And S.CDI = S.CDA. Also S.DIC = S.BDI = S.BDA.

Cor. 3. *If ever so many forces acting against one another, are kept in equilibrio, by these actions; they may be all reduced to two equal and opposite ones.*

For any two forces may, by composition, be reduced to one force acting in the same plane. And this last force, and any other, may likewise be reduced to one force acting in the plane of these; and so on, till they all be reduced at last to the action of two equal and opposite ones.

PROP. IX.

If a body impinges or acts against any plain surface; it exerts its force in a line perpendicular to that surface. 3.

Let the body A moving in direction AB, with a given velocity, impinge on the smooth plain FG at the

Fig. 3. the point B. Draw AC parallel, and BC perpendicular to FG; and let AB reprefent the force of the moving body. The force AB is, by the refolution of forces, equivalent to AC and CB. The force AC is parallel to the plain, and therefore has no effect upon it; and therefore the furface FG is only acted upon by the force CB, in a direction perpendicular to the furface FG.

Cor. 1. *If a body impinges upon another body with a given velocity; the quantity of the ftroke is as the fine of the angle of incidence.*

For the abfolute force is AB, and the force acting on the furface FG is CB. But AB : CB :: rad : S.CAB or ABF.

Cor. 2. *If an elaftic body* A *impinges upon a hard or elaftic plane* FG; *the angle of reflexion will be equal to the angle of incidence.*

For if AD be parallel to FG; the motion of A in direction AD parallel to the plane, is not at all changed by the ftroke. And by the elafticity of one or both bodies, the body A is reflected back to AD in the fame time it moved from A to B; let it pafs to D; then will AC = CD, being defcribed in equal times; confequently the angle ABC = angle CBD; and therefore the angle DBG = angle ABF.

Cor. 3. *If a non-elaftic body ftrikes another non-elaftic body; it lofes but half the motion, that it would lofe, if the bodies were elaftic.*

For non-elaftic bodies only ftop, without receding from one another; but elaftic bodies recede with the fame velocity.

PROP.

PROP. X.

The sum of the motions of two or more bodies, in any direction towards the same part, cannot be changed by any action of the bodies upon each other.

Here I reckon progressive motions affirmative; and regressive ones negative, and to be deducted out of the rest to get the sum.

1. If two bodies move the same way; since action and re-action are equal and contrary, what one body gains the other loses; and the sum remains the same as before. And the case is the same, if there were more bodies.

2. If bodies strike one another obliquely; they will act on one another in a line perpendicular to the surface acted on. And therefore by the law of action and re-action there is no change made in that direction.

3. And in a direction parallel to the striking surface, there is no action of the bodies, therefore the motion remains the same in that direction. Whence the motions will remain the same in any one line of direction.

Cor. 1. *Motion can neither be increased nor decreased, considered in any one direction; but must remain invariably the same for ever.*

This follows plainly from this Prop. for what motion is gained to one (by addition), is lost to another body (by subtraction); and so the total sum remains the same as before.

SCHOLIUM.

This Prop. does not include or meddle with such motions as are estimated in all directions. For upon this supposition, motion may be increased or decreased

Fig. decreased an infinite number of ways. For example, if two equal and non-elastic bodies, with equal velocities, meet one another; both their motions are destroyed by the stroke. Here at the beginning of the motion, they had both of them a certain quantity of motion, but to be taken in contrary directions; but after the stroke they had none.

PROP. XI.

The motion of bodies included in a given space, is the same, whether that space stands still; or moves uniformly in a right line.

For if a body be moving in a right line, and any force be equally impressed both upon the body and the line it moves in; this will make no alteration in the motion of the body along the right line. And for the same reason, the motions of any number of bodies in their several directions will still remain the same; and their motions among themselves will continue the same, whether the including space is at rest, or moves uniformly forward. And since the motions of the bodies among themselves; that is, their relative motions remain the same, whether the space including them be at rest, or has any uniform motion. Therefore their mutual actions upon one another, must also remain the same in both cases.

SECT.

Fig.

SECT. II.

The perpendicular descent of heavy bodies, their descent upon inclined Planes, and in Curve Surfaces. The Motion of Pendulums.

PROP. XII.

THE velocities of bodies, falling freely by their own weight, are as the times of their falling from rest.

For since the force of gravity is the same in all places near the earth's surface (by Post. 3.), and this is the force by which bodies descend. Therefore the falling body is urged by a force which acts constantly and equally; and therefore (by Cor. 3. Prop. V.) the velocity generated in the falling body in any time, is as the time of falling.

Cor. 1. *If a body be thrown directly upwards with the same velocity it falls with; it will lose all its motion in the same time.*

For the same active force will destroy as much motion in any time, as it can generate in the same time.

Cor. 2. *Bodies descending or ascending gain or lose equal velocities in equal times.*

PROP. XIII.

In bodies falling freely by their own weight; the spaces described in falling from rest, are as the squares of the times of falling.

For since gravity is supposed to be the same in all places near the earth. Therefore the falling

C body

Fig. body will be acted on by a force which is constantly the same; and therefore (by Prop. VI.) the spaces described, from the beginning of the motion, or since their falling from rest, will be as the squares of the times of falling.

Cor. 1. *The spaces described by falling bodies are as the squares of the velocities.*

For (Prop. XII.) the velocities are as the times of falling.

Cor. 2. *The spaces described by falling bodies, are in the compound ratio of the times, and the velocities acquired by falling.*

Cor. 3. *If a body falls through any space, and move afterwards with the velocity gained in falling; it will describe twice that space in the time of its falling.*

Cor. 4. *A body projected upward with the velocity it gained in falling, will ascend to the same height it fell from.*

SCHOLIUM.

All these things would be true if it was not for the resistance of the air, which will retard their motion a little. In very swift motions, the resistance of the air has a very great effect in destroying the motions of bodies.

PROP. XIV.

4. *If a heavy body* W *be sustained upon an inclined plane* AC, *by a power* F *acting in direction* WF *parallel to the plane; and if* AB *be parallel to the horizon and* BC *perp. to it. Then if the length* AC *denotes the weight of the body, the hight* CB *will denote the power at* F *which sustains it, and the base* AB *the pressure against the plane.*

Draw BD perpendicular to AC, then CB will be the direction of gravity, DC parallel to WF will be be the direction of the force at F, and DB the direction of the preſſure (by Prop. IX.). Therefore (by Prop. VIII.) the weight of the body, the power at F, and the preſſure; will be reſpectively as BC, CD and DB. But the triangles ABC, BDC are ſimilar, and therefore BC, CD and DB are reſpectively as AC, CB and BA. Therefore the weight, power and preſſure, are as the lines AC, CB and AB. Fig. 4.

Cor. 1. *The weight of the body, the power* F *that ſuſtains it on the plane, and the preſſure againſt the plane; are reſpectively as radius, the ſine and coſine of the planes elevation above the horizon.*

For AC, CB and AB are to one another, as radius, ſine of CAB, and ſine of ACB.

Cor. 2. *The power that urges a body* W *down the inclined plane is* $= \frac{CB}{AC} \times$ *weight of* W. *Hence,*

Cor. 3. *If a priſmatic body whoſe length is* AC *lie upon the inclined plane* AC; *it is urged down the plane with a force equal to the weight of the priſm of the length* CB.

Cor. 4. *If there be two planes of the ſame hight, and two bodies be laid on them proportional to the lengths of theſe planes; the tendency down the planes will be equal in both bodies.*

PROP. XV.

If AC *be an inclined plane,* AB *the horizon,* BC *perp. to* AB. *And if* W *be a heavy body upon the plane, which is kept there by the power* P *acting in direction* WP. *Draw* BDE *perp. to* WP. *Then the weight* W, *the power* P, *the preſſure againſt the plane; will be reſpectively as* AB, DB *and* AD. 5.

For AB being a horizontal plane is perpendicular to the action of gravity; and BD is perp. to

C 2 the

Fig. 5. the direction of the power P; and AD is the plane, which is perp. to the direction of the pressure against that plane. Therefore (Cor. 1. Prop. VIII.) the weight of the body, the power P, and the pressure; are as AB, BD and AD. And if the direction of the power WP be under the plane, the proportion will be the same, as long as BD is perpendicular to WP.

Cor. 1. *The weight of the body* W, *is to the power* P *that sustains it* : : *as cosine of the angle of traction* CWP : *to the sine of the plane's elevation* CAB.

For the weight : power : : AB : BD : : S.ADB or WDE : S.BAD : : cos DWE or CWP : S.BAC, where the angle CWP made by the plane and direction of the power is called the angle of Traction.

Cor. 2. *Hence it is the same thing as to the power and weight, whether the line of direction is above or below the plane, provided the angle of traction be the same. For an equal power will sustain the weight in both cases.*

Cor. 3. *The weight of the body* : *is to the pressure against the plane* : : *as the cosine of the angle of traction* CWP : *to the cosine of* BNP, *the direction of the power above the horizon.*

For the weight W : pressure : : AB : AD : : S.ADB : S.ABD : : S.ADE : : S.NBE : : cos. EWD or PWC : cos. BNP.

Cor. 4. *Hence the pressure against the plane is greater when the direction of the power is below the plane, the weight remaining the same.*

SCHOLIUM.

Altho' the power has the same proportion to the weight, when the angle of traction is the same; whether the direction of the power be above or below

below the plane. Yet, since the pressure upon the plane is greater, when the line of direction is below the plane. Therefore in practice, when a weight is to be drawn up hill, if it is to be done by a power whose direction is below the plane, the greater pressure in this case will make the carriage sink deeper into the earth, &c. and for that reason will require a greater power to draw it up, than when the line of direction is above the plane. Fig. 5.

PROP. XVI.

If a weight W *upon an inclined plane* AC, *be in equilibrio with another weight* P *hanging freely; then if they be set a moving, their perpendicular velocities in that place, will be reciprocally as their quantities of matter.* 6.

Take WA a very small line upon the plane AC; draw AB parallel to the horizon, and BC perp to it. Draw AF, and WR, BE perpendicular to it; and WT, DV perp. to AB. Let W descend thro' the small line WA upon the plane, then P will ascend a hight equal to AR perpendicularly; and WT will be the perpendicular descent of W. The triangles AWR and ADE are similar; and likewise the triangles AWT and ADV. Therefore WT : DV : : AW : : WD : : WR : DE. And alternately, WT : WR : : DV : DE; and WR : AR : : DE : AE; therefore WT : AR : : DV : AE : : (by the similar triangles DBV and AEB) DB : AB : : (Prop. XV.) power P : weight W.

Cor. 1. *If any two bodies be in equilibrio upon two inclined planes; their perpendicular velocities will be reciprocally as the bodies.*

Cor. 2. *If two bodies sustain each other in equilibrio, on any planes; the product of one body × by its perp.*

Fig. 6. *perp. velocity, is equal to the product of the other body × by its perp. velocity.*

PROP. XVII.

7. *If a heavy body runs down an inclined plane* CA; *the velocity it acquires in any time, moving from rest; is to the velocity acquired by a body falling perpendicularly in the same time; as the hight of the plane* CB, *to its length* CA.

The force by which a body endeavours to descend on an inclined plane, is to its weight or the force of gravity; as CB to CA (by Prop. XIV.). And as these forces always remain the same, therefore (Cor. 2. Prop. V.) the velocities generated will be as these forces, and the times of acting, directly; and the bodies reciprocally. And since the times of acting, and the bodies are the same in both cases, the velocities generated will be as these forces; that is, as the hight of the plane CB to its length CA.

Cor. 1. *The velocity acquired by a body running down an inclined plane, is as the time of its moving from rest.*

Cor. 2. *If a body is thrown up an inclined plane, with the velocity it acquired in descending; it will lose all its motion in the same time.*

PROP. XVIII.

7. *If a heavy body descends down an inclined plane* CA; *the space it describes from the beginning of the motion, is to the space described by a body falling perpendicularly in the same time; as the hight of the plane* CB, *to its length* CA.

For the force urging the body down the plane is to the force of its gravity, as CB to CA (by Prop. XIV.),

XIV.), which forces remain constantly the same. Fig.
And since (Prop. XVII.) the velocities generated 7.
in equal times on the plane, and in the perpendicular, are constantly as CB to CA; the small particles of space described with these velocities, in all the infinitely small portions of time, will still be in the same ratio; and therefore the sums of all these small spaces, or the whole spaces described from the beginning, will be in the same constant ratio of CB to CA.

Cor. 1. *The space described by a body falling down an inclined plane, in a given time, is as the sine of the plane's elevation.*

For if CB be given, and also the perp. descent; that space will be reciprocally as CA, or directly as S.CAB.

Cor. 2. *The spaces described by a body descending from rest, down an inclined plane, are as the squares of the times.*

Cor. 3. *If* BD *be drawn perp. to the plane* CA; *then in the time a body falls perpendicularly thro' the hight* CB, *another body will descend thro' the space* CD *upon the plane.*

For by similar triangles CA : CB : : CB : CD.

PROP. XIX.

If AC *is an inclined plane, the time of a body's descending thro' the plane* CA, *is to the time of falling perpendicularly thro' the hight of the plane* CB, *as the length of the plane* CA *to the hight* CB. 7

For if BD is perp. to CA, then (Cor. 2. Prop. XVIII.) space CD : space CA : : square of the time in CD : square of the time in CA : : that is, as the square of the time of descending perpendicularly

Fig. 7. cularly in CB (Cor 3. laſt) : ſquare of the time in CA. But CD : CA : : $CB^2 : CA^2$. Therefore $CB^2 : CA^2$: ſquare of the time in CB : ſquare of the time in CA. And CB : CA : : time in CB : time in CA.

Cor. 1. *If a body be thrown upwards on the plane with the velocity acquired in deſcending; it will in an equal time aſcend to the ſame hight.*

Cor. 2 *The times wherein different planes, of the ſame hight, are paſſed over; are as the lengths of the planes.*

Let the planes be CA, CF. Then time in AC : time in CB : : CA : CB. And the time in CB : time in CF : : CB : CF. Therefore *ex equo*, time in AC : time in CF : : CA : CF.

PROP. XX.

8. *If a body falls down an inclined plane, it acquires the ſame velocity as a body falling perpendicularly thro' the hight of the plane.*

Let the body run down the plane CA whoſe hight is CB. Draw DF parallel to AB, and infinitely near it. Then the velocities in DA and FB, may be looked upon as uniform. Now (Prop. XIX.) the times of deſcribing CA and CB, will be as CA and CB. Likewiſe the times of deſcribing CD and CF, will be as CD and CF; that is, as CA and CB. And by diviſion, the difference of the times, or the times of deſcribing DA and FB, will alſo be as CA and CB; that is, as DA and FB. But (from Prop. III.) the velocities are equal when the ſpaces are as the times of deſcription. Therefore velocity at A is equal to the velocity at B.

Cor.

Cor. 1. *The velocities acquired, by bodies descending on any planes, from the same hight to the same horizontal line, are equal.* Fig. 8.

Cor. 2. *If the velocities be equal at any two equal altitudes* D, F; *they will be equal at all other equal altitudes* A, B.

Cor. 3. *Hence also, if several bodies be moving in different directions, thro' any space contained between two parallel planes; and be acted on by any force, which is equal at equal distances from either plane. Then if their velocities be equal at entering that space; they will also be equal at emerging out of it.*

For dividing that space into infinitely small parts by parallel planes. Then the force between any two planes may be supposed uniform; and supposing DF, AB to represent two of these planes, then (by Cor. 2.) the velocities at D and F being equal; the velocities at A and B will be equal; that is, the velocities at entering the first part of space being equal, the velocities at emerging out of it, or at entering the second space will be equal. And for the same reason the velocities at entering the second space being equal, those at emerging out of it into the third, will be equal. And consequently the velocities at entering into, and emerging out of the third, fourth, fifth, &c. to the last will be equal respectively.

PROP. XXI.

If a body falls from the same hight, thro' any number of contiguous planes AB, BC, CD; *it will at last gain the same velocity as a body falling perpendicularly from the same hight.* 9.

Let FH be a horizontal line, FD perp. to it. Produce the planes BC, DC to G and H. Then

(Cor.

Fig. 9. (Cor. 1. Prop. XX.) the velocity at B is the same whether the body descend thro' AB or GB. And therefore the velocity at C will be the same, whether the body descends thro' ABC or thro' GC, and this is the same as if it had descended thro' HC. And consequently it will have the same velocity at D, in descending thro' the planes ABCD, as in descending thro' the plane HD; that is, (Prop. XX.) as it has in descending thro' the perpendicular FD.

Cor. 1. *Hence a body descending along any curve surface, will acquire the same velocity, as if it fell perpendicularly thro' the same hight.*

For let the number of planes be increased, and their length diminished ad infinitum, and then ABCD will become a curve. And the velocity acquired by descending thro' these infinite planes; that is, thro' the curve ABCD, will be the same as in falling perpendicularly thro' FD.

Cor. 2. *If a body descends in a curve, and another descends perpendicularly from the same hight. Their velocities will be equal at all equal altitudes.*

Cor. 3. *If a body, after its descent in a curve, should be directed upwards with the velocity it had gained; it will ascend to the same hight from which it fell.*

For since gravity acts with the same force whether the body ascends or descends, it will destroy the velocity in the ascent, in the same time it did generate it in the descent.

Cor. 4. *The velocity of a body descending in any curve, is as the square root of the hight fallen from.*

For it is the same as in falling perpendicularly; and in falling perpendicularly, it is as the square root of the hight.

Cor.

Cor. 5. *If a body, in moving thro' any ſpace* ED, *be acted on uniformly by any force; its velocity at emerging out of it at* D, *will be equal to the ſquare root of the ſum of the ſquares of the velocity at* E *in entering of it, and of the velocity acquired in falling from reſt thro' that ſpace* ED. *And this holds whether the body moves perpendicularly or obliquely.* Fig. 9.

For let the body enter the ſpace ED at E, with the velocity acquired in falling thro' FE. Then (Prop. XIII. Cor. 1.) the ſquare of the velocity at E will be as FE; and the ſquare of the velocity at D, as FD; and the ſquare of the velocity at D falling from E, will be as ED. But FD = FE + ED; therefore the ſquare of the velocity at D (falling thro' FD) = ſquare of the velocity at E + ſquare of the velocity at D (falling thro' ED). And (Cor. 1. of this Prop.) the velocity will be the ſame whether the body deſcends perpendicularly or obliquely.

PROP. XXII.

The times of bodies deſcending thro' two ſimilar parts of ſimilar curves, placed alike, are as the ſquare roots of their lengths. 10.

Let ABCD and *abcd* be two ſimilar curves, and ſuppoſe BC and *bc* to be infinitely ſmall, and ſimilar to the whole; that is, ſo that BC : *bc* : : AB : *ab*. Draw FA parallel to the horizon, and HB, *hb* perp. to it. Then if two bodies deſcend from A and *a* (Cor. 4. Prop. XXI.) the velocities at B and *b* will be as $\surd$HB and $\surd hb$; that is, as $\surd$AB and $\surd ab$, becauſe AB, *ab* are ſimilar parts. Therefore (Prop. III. Cor. 1.) the times of deſcribing BC and *bc*, are as $\frac{\mathrm{BC}}{\surd \mathrm{AB}}$ and $\frac{bc}{\surd ab}$; that is,

as

Fig. 10. as $\frac{AB}{\sqrt{AB}}$ and $\frac{ab}{\sqrt{ab}}$ or as $\sqrt{AB}$ and $\sqrt{ab}$; that is, as $\sqrt{AD}$ and $\sqrt{ad}$, becauſe the curves are ſimilarly divided in B and *b*. After the ſame manner the times of deſcribing any other two ſimilar parts as BC, *bc*, will be as $\sqrt{AD}$ and $\sqrt{ad}$. Therefore by compoſition the times of deſcribing all the BC's, and all the *bc*'s will be as $\sqrt{AD}$ and $\sqrt{ad}$. That is, the time of deſcribing the curve AD to the time of deſcribing the curve *ad*, is as $\sqrt{AD}$ to $\sqrt{ad}$.

Cor. *If two bodies deſcend thro' two ſimilar curves* ABD, *and abd; the axes of the curves* FD, F*d are as the ſquares of the times of their deſcending.*

For $\sqrt{FD} : \sqrt{Fd} :: \sqrt{AD} : \sqrt{ad}$ :: time of deſcending thro' ABD : time of deſcending thro' *abd*. And FD, F*d*, are as the ſquares of the times.

PROP. XXIII.

11. *A body will deſcend thro' any chord of a circle, in the ſame time that another deſcends perpendicularly thro' the diameter.*

Draw the diameter AB perpendicular to the horizon, and the cords CA, CB. Then ſince BC is perpendicular to AC, therefore (Prop. XVIII. Cor. 3.) the time of deſcending thro' the cord AC is equal to the time of falling thro' AB.

Draw CD parallel to AB, and DB parallel to CA, then is CD equal to AB. And by reaſon of the parallels, the angle DBC = angle BCA = a right angle. Then ſince DB is perp. to CB, therefore (Cor. 3. Prop. XVIII.) a body will deſcend thro' the inclined plane CB, in the ſame time that another falls thro' CD, or which is the ſame thing, thro' its equal AB.

Cor.

Cor. 1. *Hence the times of descending thro' all the cords of a circle drawn from* A *or* B, *are equal among themselves.* Fig. 11.

Cor. 2. *The velocity gained by falling thro' the cord* CB, *is as its length* CB.

For the velocity gained in falling thro' CB is the same as is gained by falling thro' EB; and that velocity is to the velocity gained by falling thro' AB, as $\sqrt{BE}$ to $\sqrt{AB}$ (by Cor. 1. Prop. XIII.); that is, as BC to BA. Therefore if the given velocity in falling down AB be represented by AB. The velocity gained in falling down CB will be represented by CB; and so that in any other cord, by its length.

PROP. XXIV.

If a pendulum vibrates in the small arch of a circle; the time of one vibration, is to the time of a body's falling perpendicularly thro' half the length of the pendulum; as the circumference of a circle, to the diameter. 12

If a pendulum suspended by a thread, &c. be made to vibrate in any curve; it is the same thing as if it descended down a smooth polished body made in the form of that curve. For the motions, velocities, and times of moving will be the same in both.

Let OD or OE be the pendulum vibrating in the arch ADC, whose radius is OD. Let OD be perp. to the horizon, and take the arch E*e* infinitely small, and draw ABC, EFG, *efg*, perp. to OD; and draw the cord AD. About BD describe the semicircle BGD. Draw *er* and G*s* perp. to EG.

Put t = time of descending thro' the diameter 2OD, or thro' the cord AD. Then the velocities gained

Fig. 12. gained by falling thro' 2OD, and by the pendulum's descending thro' the arch AE, will be as $\sqrt{2OD}$ and $\sqrt{BF}$. And the space described in the time t, after the fall thro' 2OD, is 4OD. But the times are as the spaces, divided by the velocities. Therefore, $\frac{4OD}{\sqrt{2OD}}$ or $2\sqrt{2OD} : t$ (time of its description) $:: \frac{Ee}{\sqrt{BF}}$: time of describing $Ee = \frac{t \times Ee}{2\sqrt{2OD \times BF}}$.

But by the similar triangles OEF, Eer; and KGF, Ggs; we shall have $\frac{EF}{OD} \times Ee = er = Ff = Gs = \frac{FG}{KD} \times Gg$. Whence $Ee = \frac{OD \times FG}{KD \times EF} \times Gg$. Therefore the time of describing $Ee = \frac{t \times OD \times FG \times Gg}{2KD \times EF \sqrt{BF \times 2OD}} = \frac{t \times OD \times \sqrt{BF \times FD} \times Gg}{2KD \sqrt{BF} \times \sqrt{DO + OF \times FD} \times \sqrt{2OD}} = \frac{t \times \sqrt{OD} \times Gg}{2KD \times \sqrt{DO + OF} \times \sqrt{2}} = \frac{t \times \sqrt{2OD}}{4KD \times \sqrt{DO + OF}} \times Gg. = \frac{t \times \sqrt{2OD}}{2BD \times \sqrt{2OD - DF}} \times Gg$. But DF, in its mean quantity for all the arches Gg, is nearly equal to DK. Therefore the time of describing $Ee = \frac{t \times \sqrt{2OD}}{2BD \sqrt{2OD - DK}} \times Gg$. Whence the time of describing the arch AED $= \frac{t \times \sqrt{2OD}}{2BD \sqrt{2OD - DK}} \times BGD$. And the time of describing the whole arch ADC, or the time of one oscillation is $=$

$t \times$

Fig. 12.

$\dfrac{t \times \sqrt{2OD}}{2BD\sqrt{2OD - DK}} \times 2BGD$. But when the arch ADC is exceeding ſmall, DK vaniſhes, and then the time of oſcillation in a very ſmall arch is $=$ $\dfrac{t \times \sqrt{2OD}}{2BD\sqrt{2OD}} \times 2BGD = \frac{1}{2} t \times \dfrac{2BGD}{BD}$. But if t be the time of deſcending thro' 2OD, $\frac{1}{2} t$ is the time of deſcending thro' $\frac{1}{2}$ OD. And therefore BD the diameter, is to 2BGD the circumference; as the time of falling thro' half the length of the pendulum, to the time of one vibration.

Cor. 1. *In a ſmall arch* AED, *the time of deſcending thro' the cord* AD, *is to the time of deſcending thro' the arch* AED; *as the diameter* BD, *to* $\frac{1}{4}$ *the circumference.*

For the time of deſcending thro' the arch AED $= t \times \dfrac{BGD}{2BD}$; therefore BD : $\frac{1}{2}$ BGD : : t : time in AED.

Cor. 2. *All the vibrations of the ſame pendulum, in arches not very large, are performed nearly in the ſame time.*

Cor. 3. *If* KD *be biſſected in* L, *and* T *be* $=$ *time of vibration in a very ſmall arch. Then* $T + \dfrac{KL}{DO + OK} \times T$ *will be the time of vibration in any arch* ADC, *nearly.*

For we found the time of vibration in ADC $=$ $\dfrac{t \times BGD}{BD} \times \sqrt{\dfrac{2OD}{2OD - DK}} = T \times \sqrt{\dfrac{2OD}{OD + OK}}$; and the three lines DO + OK, DO + OL, and DO + OD are in arithmetical progreſſion; but ſince KD is very ſmall, they are nearly in geometri-

cal

Fig. 12. cal progreſſion; whence $\sqrt{\frac{2OD}{DO+OK}} = \frac{DO+OL}{DO+OK}$.

Therefore the time of vibration $= T \times \frac{DO+OL}{DO+OK}$

$= T \times \frac{DO+OK+KL}{DO+OK} = T + T \times \frac{KL}{DO+OK}$.

Cor. 4. *Hence a falling body will deſcend thro' a ſpace of 16 feet, and 1 inch, in a ſecond of time.*

For by obſervation, a pendulum 39.13 inches long will ſwing ſeconds. And $t \times \frac{BGD}{BD} = 1$ ſecond, and $\frac{BD}{BGD} = t$, or $\frac{2}{3.1416} =$ time of falling thro' 2×39.13. Whence (Prop. XIII.) $\frac{4}{\overline{3.1416}^2}$: $2 \times 39.13 :: 1^2 : \frac{39.13}{2} \times \overline{3.1416}^2 = 193.096$ inches $= 16.09$ feet.

PROP. XXV.

The lengths of two pendulums, deſcribing ſimilar arches, are as the ſquares of the times of vibration.

For (Prop. XXII.) the times of deſcending thro' two ſimilar curves, are as the ſquare roots of the lengths of the curves; that is, as the ſquare roots of the lengths of the pendulum, their centers being alike ſituated. Therefore the lengths of the pendulums are as the ſquares of the times of vibrating.

Cor. 1. *The times of vibration of pendulums in ſmall arches of circles, are as the ſquare roots of the lengths of the pendulums.*

For if the arches are ſimilar, the times of vibration are in that proportion. And (Prop. XXIV.

Cor.

Cor. 2.) if the arches are small, tho' not similar, the vibrations will be the same as before. Fi

Cor. 2. *The velocity of a pendulum at its lowest point, is as the cord of the arch it descends thro'.*

For the velocity at the lowest point is equal to the velocity gained in descending thro' the cord; for they are both of them the same as a body acquires by falling thro' their common altitude. And (Prop. XXIII. Cor. 2.) the velocity gained in falling thro' the cord, is as the length of the cord. Therefore the velocities of a pendulum in different arches, are in the same ratio.

PROP. XXVI.

Pendulums of the same length vibrate in the same time, whether they be heavier or lighter.

For let the two pendulums P, p, be of the same length; they will each of them fall thro' half the length of the pendulum in the same time. For (Cor. 2. Prop. V.) the velocity generated in any given time, is as the force directly and matter reciprocally. But in the two pendulums, the forces that generate their motions, are their weights, which are as their quantities of matter. Whence we have the velocity of P, to the velocity of p; as $\frac{P}{P}$ to $\frac{p}{p}$, or as 1 to 1; and therefore equal velocities are generated in the same time. Consequently, equal spaces will be described in the same time, and therefore they will fall thro' half the length of one of them in an equal time. And therefore (Prop. XXIV.) their vibrations will be performed in the same time.

Cor. *Hence all bodies whether greater or lesser, heavier or lighter, near the earth's surface will fall*

Fig. *thro' equal spaces in equal times; abating the resistance of the air.*

Because they are as much retarded by their matter, as accelerated by their weight. The weight and the matter being exactly proportional to one another.

PROP. XXVII.

The lengths of pendulum's vibrating in the same time, in different places of the world, will be as the forces of gravity.

For (by Prop. V. Cor. 2.) the velocity generated in any time is as the force of gravity directly, and the quantity of matter reciprocally. And the matter being supposed the same in both pendulums, the velocity is as the force of gravity; and the space passed thro' in a given time, will be as the velocity; that is, as the gravity. Therefore if any two spaces be descended thro' in any time, and two pendulums be made, whose lengths are double these spaces; these pendulums (by Prop. XXIV.) will vibrate in equal times; therefore the lengths of the pendulums, being as the spaces fallen thro' in equal times, will be as the forces of gravity.

Cor. 1. *The times wherein pendulums of the same length will vibrate, by different forces of gravity; are reciprocally as the square roots of the forces.*

For (Cor. 2. Prop. V.) when the matter is given, the velocity generated is as the force × by the time. And (Prop. VI.) the space descended thro' by any force, is as the force and square of the time. Let these spaces be the lengths of the pendulums, then the lengths of the pendulums are as the forces and the squares of the times of falling thro' them. But (Prop. XXIV.) the times of falling thro' them are in a given ratio to the times of vibration; whence

whence the lengths of pendulums are as the forces and the ſquares of the times of vibration; therefore when the lengths are given, the forces will be reciprocally as the ſquares of the times; and the times of vibration reciprocally as the ſquare roots of the forces.

Cor. 2. *The lengths of pendulums in different places, are as the forces of gravity, and the ſquares of the times of vibration.*

This is proved under Cor. 2. Hence,

Cor. 3. *The times wherein pendulums of any length, perform their oſcillations; are as the ſquare roots of their lengths directly, and the ſquare roots of the gravitating forces reciprocally.*

Cor. 4. *The forces of gravity in different places, are as the lengths of pendulums directly, and the ſquares of the times of vibration reciprocally.*

PROP. XXVIII. *Prob.*

To find the length of a pendulum, that ſhall make any number of vibrations in a given time.

Reduce the given time into ſeconds, then ſay, as the ſquare of the number of vibrations given : to the ſquare of this number of ſeconds : : ſo is 39.13 : to the length of the pendulum ſought, in inches.

Ex. Suppoſe it makes 50 vibrations in a minute, here a minute is = 60 ſeconds; then,

As 2500 (the ſquare of 50) : 3600 (the ſquare of 60) : : 39.13 : to the length $= \frac{3600 \times 39.13}{2500}$ $= \frac{140868}{2500} = 56.34$ inches, the length required.

If it be required to find a pendulum that ſhall vibrate ſuch a number of times in a minute; you need only divide 140868, by the ſquare of the number of vibrations given, and the quotient will be the length of the pendulum.

This practice is deduced from Prop. XXV. for let p be the length of the pendulum, n the number of vibrations, t the time they are to be performed in. Then $39.13 : 1^2 :: p : \frac{p}{39.13} =$ ſquare of the time of one vibration, and $\sqrt{\frac{p}{39.13}} =$ time of one vibration; then if t be divided by $\sqrt{\frac{p}{39.13}}$ it will give n; that is, $t\sqrt{\frac{39.13}{p}} = n$, whence $tt \times 39.13 = nnp$, and $nn : tt :: 39.13 : p$.

If the pendulum is a thread with a little ball at it, then the diſtance between the point of ſuſpenſion and the center of the ball is eſteemed the length of the pendulum. But if the ball be large, ſay as the diſtance between the point of ſuſpenſion, and the center of the ball, is to the radius of the ball; ſo the radius of the ball to a third proportional. Set $\frac{2}{5}$ of this from the center of the ball downward, gives the center of oſcillation. Then the whole diſtance from the point of ſuſpenſion to this center of oſcillation, is the true length of the pendulum.

13. If the bob of the pendulum be not a whole ſphere, but a thin ſegment of a ſphere, as AB, as in moſt clocks; then to find the center of oſcillation, ſay as the diſtance between the point of ſuſpenſion, and the middle of the bob, is to half the breadth of the bob; ſo half the breadth of the bob, to a third proportional. Set one third of this length from the middle of the bob downwards, gives the center of oſcillation. Then the diſtance between

between the centers of suspension and oscillation, is Fig the exact length of the pendulum.

PROP. XXIX. *Prob.*

Having the length of a pendulum given; to find how many vibrations it shall make, in any given time.

Reduce the time given into seconds, and the pendulum's length into inches; then say, as the given length of the pendulum : to 39.13 : : so is the square of the time given : to the square of the number of vibrations, whose square root is the number sought.

Example. Suppose the length of the pendulum is 56.34 inches, to find how often it will vibrate in a minute.

1 minute = 60 seconds. Then, 56.34 (the length of the pendulum) : 39.13 : : 3600 (the square of 60) : to the square of the number of vibrations $= \frac{3600 \times 39.13}{56.34} = \frac{140868}{56.34} = 2500$, and $\sqrt{2500} = 50 =$ the number of vibrations sought.

If the time given be a minute, you need only divide 140868 by the length, and extract the root of the quotient for the number of vibrations.

This is the reverse of the last problem, therefore supposing as before in that problem, we have $tt \times 39.13 = nnp$; therefore $p : 39.13 : : tt : nn$.

They that would see a further account of the motions of bodies upon inclined planes, the vibrations of pendulums, and the motion of projectiles; may consult my large book of Mechanics, where they will meet with full satisfaction.

SECT. III.

Of the Center of Gravity; the equilibrium of beams of timber; the directions and quantities of the forces neceſſary to ſuſtain them.

PROP. XXX.

A body cannot deſcend or fall downwards, except only when it is in ſuch a poſition, that by its motion, its center of gravity deſcends.

4. Let the body A ſtand upon the horizontal plane BK, and let C be its center of gravity; draw CD perpendicular to the plane BK. And let the body be ſuſpended at the point C, upon the perpendicular line CD. Then (def. 12.) it will remain unmoved upon the line CD. And as CD is perp. to the horizon, it has no inclination to move one way more than another, therefore it will move no way but remain at reſt. Take away the line CD, and let the body be ſupported by the line BC; if BC be fixed, the body will remain at reſt on the line CB But if CB be movable about B, the body ſuſpended at C, will endeavour to move with its center of gravity downwards along the arch CE, about B as a center, towards N. And for the ſame reaſon the body will endeavour to fall the contrary way, moving about the point N; I ſay, this will be the caſe when CD is ſituated between B and N. But theſe two motions being contrary to one another, will hinder each other's effects; and the body will be ſuſtained without falling.

Again,

Again, let the body F be suspended with its center of gravity I upon the perpendicular IH. As this line has no inclination to move to any side, it will therefore remain at rest. Take away the line IH, and let the center of gravity I be suspended on the line IG, then the body will endeavour to descend along the arch IK, for the highest point of the arch is in the perpendicular erected at G. For the same reason if the body be suspended on the line OI, it will endeavour to descend towards K, about the center O; now as both these motions tend the same way, and there is nothing to oppose them; the body must fall towards K. In both these cases it is plain, that when the center of gravity by its motion, descends, the body will fall; but if not, the body will be supported without falling. F 1

Cor. 1. *If a body stands upon a plane, if a line be drawn from the center of gravity perpendicular to the horizon; if this line fall within the base on which the body stands, it will be supported without falling. But if the perpendicular falls without the base, the body will fall.*

For when the perpendicular falls within the base, the body can be moved no manner of way, but the center of gravity will rise. And when the perpendicular falls without the base, towards any side; if the body be moved towards that side, the center of gravity descends; and therefore the body will fall that way.

Cor. 2. *If a perpendicular drawn from the center of gravity perp. to the horizon, fall upon the extremity of the base; the body may stand, but the least force whatever, will cause it to fall that way. And the nearer the perp. is to any side, the easier it will be made to fall, or is sooner thrust over. And the nearer the perp. is to the middle of the base, the firmer the body stands.*

Cor. 3. *Hence if the center of gravity of a body be ſupported, the whole body is ſupported. And the place of the center of gravity muſt be deemed the place of the body; and is always in a line drawn perpendicular to the horizon, thro' the center of gravity.*

Cor. 4. *Hence all the natural actions of animals may be accounted for from the properties of the center of gravity.*

When a man endeavours to walk, he ſtretches out his hind leg, and bends the knee of his fore leg, by which means his body is thruſt forward, and the center of gravity of his body is moved forward beyond his feet; then to prevent his falling, he immediately takes up his hind foot, and places it forward beyond the center of gravity; then he thruſts himſelf forward, by his leg which now is the hindmoſt, till his center of gravity be beyond his fore foot, and then he ſets his hind foot forward again; and thus he continues walking as long as he pleaſes.

In ſtanding, a man having his feet cloſe together cannot ſtand ſo firmly, as when they are at ſome diſtance; for the greater the baſe, the firmer the body will ſtand; therefore a globe is eaſily moved upon a plane, and a needle cannot ſtand upon its point, any otherwiſe than by ſticking it into the plane.

When a man is ſeated in a chair, he cannot riſe till he thruſts his body forward, and draws his feet backward, till the center of gravity of his body be before his feet; or at leaſt upon them; and to prevent falling forward, he ſets one of his feet forward, and then he can ſtand, or ſtep forward as he pleaſes.

All other animals walk by the ſame rules; firſt ſetting one foot forward, that way the center of gravity leans, and then another.

In walking up hill, a man bends his body forward, that the center of gravity may lie forward of his feet; and by that means he prevents his falling backwards. Fig.

In carrying a burthen, a man always leans the contrary way that the burthen lies; so that the center of gravity of the whole of his body and the burthen, may fall upon his feet.

A fowl going over an obstacle, thrusts his head forward, by that means moving the center of gravity of his whole body forwards; so that by setting one foot upon the obstacle, he can the more easily get over it.

PROP. XXXI.

In any two bodies A, B, *the common center of gravity* C, *divides the line joining their centers, into two parts, which are reciprocally as the bodies.* AC : BC : : B : A. 15.

Let the line ACB be supposed an inflexible lever; and let the lever and bodies be suspended on the point C. Then let the bodies be made to vibrate about the immovable point C; then will A and B describe two arches of circles about the center C, and these arches will be as the velocities of the bodies, and these arches are also as the radii of the circles AC and BC. Therefore their velocities are as the radii. Whence, velocity of A : velocity of B : : AC : CB : : (by supposition) B : A. Therefore A × velocity of A = B × velocity of B. Whence (Prop. II.) the quantities of motion of the bodies A and B are equal, and (Ax. 9.) therefore they cannot move one another, but must remain at rest; and consequently (def. 12.) C is the center of gravity of A and B.

Cor.

Fig. 15. Cor. 1. *The products of each body multiplied by its distance from the common center of gravity, are equal.* CA × A = CB × B.

Cor 2. *If a weight be laid upon* C, *a point of the inflexible lever* AB, *which is supported at* A *and* B; *the pressure at* A *to the pressure at* B, *will be as* CB *to* CA.

For let the bodies A, B, be both placed in C; then (Cor. 3. Prop. XXX.) since it is the same thing whether the bodies be at A and B, or both of them at C, their center of gravity; therefore the pressures at A and B will be the same in both cases. But when they are at A and B, upon the lever ACB, their pressures are A and B, being the same with the weights; therefore when they are both at C, the pressures at A and B will still be A and B. Therefore (Cor. 1.) since it is CA × A = CB × B; therefore CA : CB : : B : A : : pressure at B : pressure at A.

PROP. XXXII.

15. *If there be three or more bodies, and if a line be drawn from one body* E *to the center of gravity of the rest* C. *Then the center of gravity of all the bodies divides the line* CD, *in two parts in* D, *which are reciprocally as the body* E *to the sum of all the other bodies.* CD : DE : : E : A + B *&c.*

For suppose AB and CE to be two inflexible lines; and let the body W = A + B &c. and let W be placed in the center of gravity C. Then by the last Prop. CD : DE : : E : W or A + B &c.

Cor. *The body* E × DE *the distance from the common center of gravity, is equal to the sum of the bodies* A + B *&c.* × *by* DC *the distance of their center from the common center of gravity.*

PROP.

PROP. XXXIII.

If A, B, *be two bodies*, C *their center of gravity.* 16
F *any point in the line* AB. *Then will* $FA \times A + FB \times B = FC \times \overline{A + B}$.

For (Cor. 1. Prop. XXXI.) $CA \times A = CB \times B$; that is, $\overline{FA - FC} \times A = \overline{FC - FB} \times B$; whence, by transposition $FA \times A + FB \times B = FC \times \overline{A + B}$.

Cor. *Hence the bodies* A *and* B *have the same force to turn the lever* AF *about the point* F, *as if they were both placed in* C *their center of gravity.*

PROP. XXXIV.

If several bodies A, B, E *&c. be placed on an* 17
inflexible streight lever; and if D *be their common center of gravity; and if* F *be any point in the line* AE, *then* $FA \times A + FB \times B + FE \times E$ *&c.* $= FD \times \overline{A + B + E}$ *&c.*

For if $A + B = W$, then $FA \times A + FB \times B + FE \times E = FC \times \overline{A + B} + FE \times E = FC \times W + FE \times E =$ (Prop. XXXIII.) $FD \times \overline{W + E} = FD \times \overline{A + B + E}$, in the three bodies A, B, E. And after the same manner, if there be four bodies, put $W = A + B + E$, and it will be proved the same way, that the sum of all the products, $FA \times A + FB \times B$ &c. $=$ distance of the common center of gravity $\times$ by all the four. And so on for more bodies.

Cor. *The same Prop. will hold good, when the bodies are not in the line* AF, *but any-where in the perpendiculars passing thro' the points* A, B, E *&c.*

PROP.

'ig.

PROP. XXXV.

7. *If there be any number of bodies* A, B, E, *&c. either placed in the line* AF, *or any way in the perpendiculars passing thro'* A, B, E. *And if* D *be the center of gravity of all the bodies; and* F *be any point in the line* AF. *Then the distance of the center of gravity* $FD = \frac{FA \times A + FB \times B + FE \times E}{A + B + E}$.

For whether the bodies be in the points A, B, E, or in the perpendiculars, it will be (by Prop. XXXIV. and Cor.) that $FA \times A + FB \times B + FE \times E = FD \times \overline{A + B + E}$. Whence $FD = \frac{A \times FA + B \times FB + E \times FE}{A + B + E} =$ sum of all the products of each body multiplied by its distance, divided by the sum of the bodies.

Cor. 1. *If a single body only was placed on the lever* AF; *then the distance of the center of gravity of that body, is equal to the sum of the products of all the particles of the body, each multiplied by its distance from a given point* F, *and divided by the body.*

For if A, B, E &c. are several particles of the body, then A + B + E &c. = the body; and $FD = \frac{A \times FA + B \times FB + E \times FE}{\text{body}}$.

Cor. 2. *If there be several bodies* A, B, E, *&c. placed upon the lever* AF. *They act with the same force in turning the lever about any given point* F, *as if they were all placed in* D *the common center of gravity of all the bodies.*

Scholium.

If any of the bodies be placed on the contrary side of F, *their respective products will be negative.*

For

For they act the contrary way in turning the le- Fig
ver about. 17

PROP. XXXVI.

If ſeveral bodies A, B, E, G, H, *be placed on* 18
the lever AH, *and* F *be the center of gravity of all the weights. Then* $FA \times A + FB \times B + FE \times E = FG \times G + FH \times H$.

For let the lever be ſuſpended on the point F, then the two ends will be in equilibrio, as F is the center of gravity. Let D be the center of gravity of A, B, E; and I the center of gravity of G, H. Then (Cor. Prop. XXXIII.) it is the ſame thing whether the bodies on one ſide be placed at A, B, E, or all of them in the point D. And whether thoſe at the other end be placed at G, H; or all of them at I. But ſince F is the center of gravity, $DF \times \overline{A + B + E} = FI \times \overline{G + H}$, and therefore $A \times AF + B \times BF + E \times EF = G \times GF + H \times HF$ (by Prop. XXXIV.)

Cor. 1. *If ſeveral bodies* A, B, E, G, H, *be placed on an inflexible lever, and if* $A \times FA + B \times FB + E \times FE = G \times FG + H \times FH$. *Then* F *is the center of gravity of all the bodies.*

For no other point will anſwer the equation.

Cor. 2. *If ſeveral bodies* A, B, E, G, H, *be placed upon a lever* AH, *or ſuſpended at theſe points by ropes; and if* $A \times FA + B \times FB + E \times FE = G \times FG + H \times FH$; *they will be in equilibrio upon the point* F.

This appears by Def. 12, and F is the center of gravity.

PROP.

Fig.

PROP. XXXVII.

19. *If a heavy body* AB, *suspended by two ropes* AC, BD, *remains at rest; a right line perpendicular to the horizon, passing thro' the intersection* F *of the ropes; will also pass thro' the center of gravity* G, *of the body.*

If AC and BD be produced to F where they intersect; then (ax. 12.) it is the same thing whether the force acting in direction AC be applied to C or F; and whether the force acting in direction BD be applied to the point D or F. Suppose then that they both act at F, and then it is the same thing, as if the body was suspended at F by the two strings AF, BF. And since the body is at rest, therefore (Ax. 7.) the body, that is, the center of gravity G, is at the lowest place it can get; and therefore is in the plumb line FG. For if the body be made to vibrate, the center of gravity G will describe an arch of a circle, of which G (being in the perp. FG) is the lowest point.

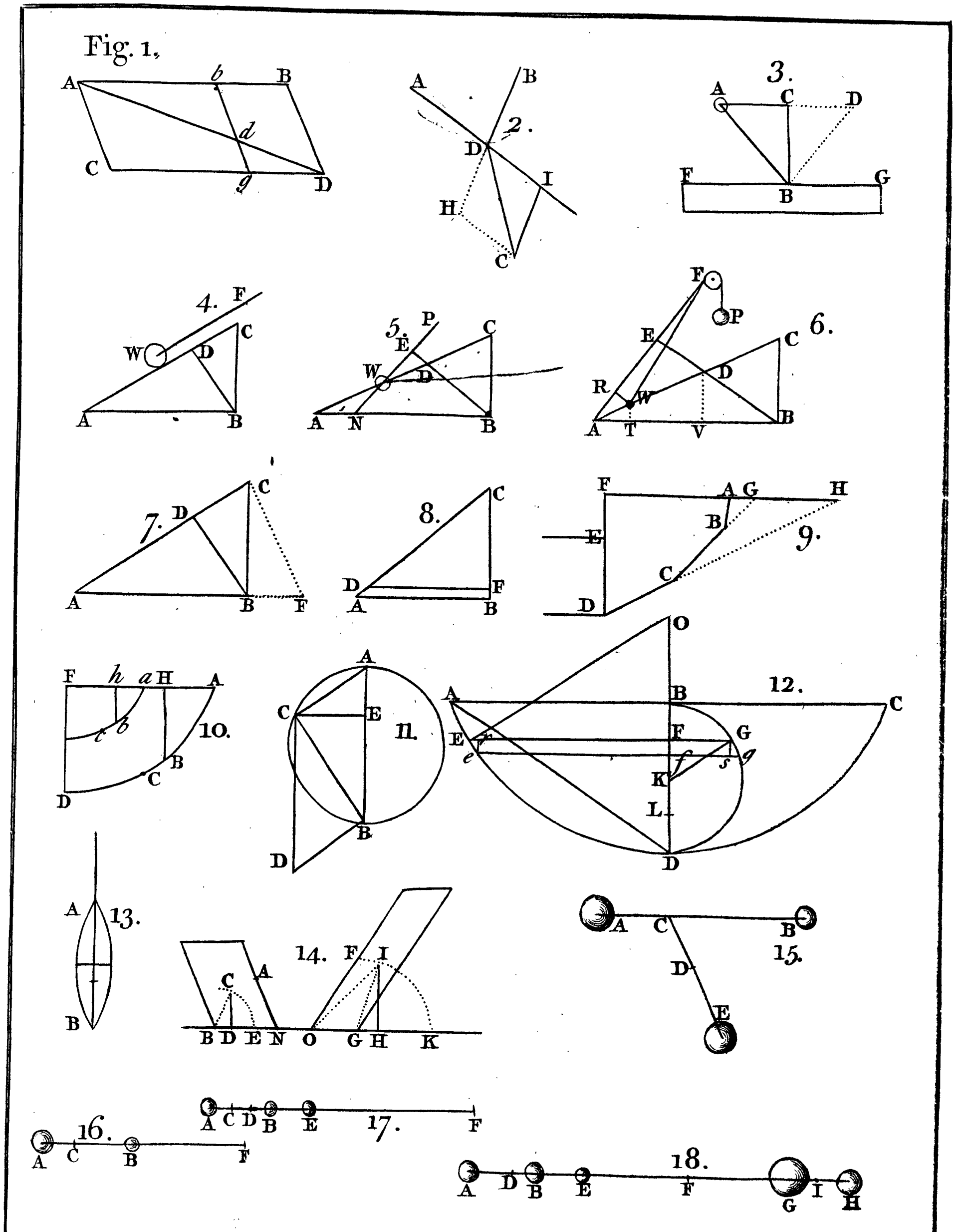
Cor. 1. *Hence if* GN *be drawn parallel to* AC; *then the weight of the body, the forces acting at* C *and* D, *are respectively as* FG, GN, *and* FN.

This is evident by Prop. VIII.

Cor. 2. *If a heavy body* AB, *be supported by two planes*, IKL, *and* EHG, *at* H and K; *and* HF *and* KF *be drawn perpendicular to these planes; and if* FG *be drawn from the intersection* F, *perp. to the horizon, it will pass thro' the center of gravity* G, *of the body.*

For since the body is sustained by these planes, therefore the planes re-act against the body (by Prop. IX.), in the directions HF, KF pependicular to these planes. Therefore it is the same thing

as



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as if the body was sustained by the two ropes HF, KF. For the directions and quantities of the forces, acting at H and K are the same in both cases. And further, if the body be made to vibrate round F, the points H, K will describe two arches of circles, coinciding with the touching planes at H, K; therefore moving the body up and down the planes, will be just the same thing, as making it vibrate in the ropes, HF and KF; and consequently, the body can rest in neither case, but when the center of gravity G is in the perpendicular FG. Fig. 19.

SCHOLIUM.

If any body should deny the truth of this Prop. or its corollaries, against the clearest force of demonstration. It lies upon them to shew where the demonstration fails, or what step or steps thereof do not hold good, or are not truly deduced from the foregoing. If they cannot do this, what other reasons they may assign, can signify nothing at all to the purpose. And if such person, ignorant of the laws of nature, and the resolution of forces, would object against this practice, of substituting planes perpendicular to the lines or cords sustaining any weights, instead of these cords. Let him first read Sir *J. Newton's* *Principia*, Cor. 2. to the laws of nature, where he will see this practice exemplified, and then make his objections.

And for the sake of such persons as understand not how to apply the method of composition and resolution of forces, I shall add a few problems to prevent their being misled by the rash judgment of some people, who having brought out false solutions to some problems by their own ill management, condemn the method as erroneous; when the fault really lies in their own ignorance, and not at all in the method itself.

PROP.

Fig.

PROP. XXXVIII. *Prob.*

20. *To determine the position of a beam* CD, *movable about the end* C, *and sustained by a given weight* Q, *hanging at a rope* QAD, *which goes over a pulley at* A *and is fixed to the other end* D.

Draw AF, CK parallel to the horizon, FDE perp. to it, and KD perp. to CD; and let G be the center of gravity, w = weight of the beam. Then if the beam was to lie horizontally (Prop. XXXI. Cor. 2.) it would be GC : GD : : pressure at D : pressure at C; and GC : CD : : pressure at D : w; whence the pressure at D $= \frac{GC}{CD} w$, horizontally. And (Prop. XIV.) CD : CE : : $\frac{GC}{CD} w$; $\frac{CE \times CG}{CD^2} w$ = pressure in direction DK. Produce AD to O, and draw OI parallel to CD. Then the beam is sustained by three forces in directions OD, DI and IO; and DI : DO : : S.IOD or ODC or ADC : rad; whence S.ADC : rad : : $\frac{CE \times CG}{CD^2} \times w$: force DO or Q $= \frac{\text{rad} \times CE \times CG}{\text{S.ADC} \times CD^2} w$. Therefore w : Q : : S.ADC : rad $\times \frac{CE \times CG}{CD^2}$: : S.ADC : $\frac{CG}{CD} \times$ S.FDC, because $\frac{\text{rad} \times CE}{CD}$ = S.EDC or FDC.

PROP. XXXIX. *Prob.*

21. *Let the beam* ED, *be sustained by the weights* P, Q, *by means of the ropes* DCP, EAQ, *going over the pulleys* C, A, *in the horizontal line* AC. *To find the position of the beam; having the weights* P, Q, *given.*

1 *Way.*

Let G be the center of gravity of the beam. Thro' D, E, draw HDS, FER perpendicular to AC.

AC: Then (Cor. 2. Prop. VIII.) S.EDS : P the tension of the thread CD : : S.CDS or CDH : $\frac{\text{S.CDH}}{\text{S.EDS}} \times$ P the tension of DE. Also, S.DER or EDS : Q : : S.AER or AEF : $\frac{\text{S.AEF}}{\text{S.EDS}} \times$ Q the tension of DE in a contrary direction. Then as the beam is in equilibrio, these forces or tensions balance one another; therefore $\frac{\text{S.CDH}}{\text{S.EDS}} \times \text{P} = \frac{\text{S.AEF}}{\text{S.EDS}} \times \text{Q}$. Then P : Q : : S.AEF : S.CDH; which may be otherwise expressed, for AE : rad : : AF : S.AEF $= \frac{\text{AF}}{\text{AE}} \times$ rad; and DC : rad : : HC : $\frac{\text{HC}}{\text{DC}} \times \text{rad} =$ S.HDC. Whence P : Q : : $\frac{\text{AF}}{\text{AE}} : \frac{\text{CH}}{\text{CD}} : : \frac{\text{AF}}{\text{HC}} : \frac{\text{AE}}{\text{DC}}$.

Fig. 21.

2 *Way*.

Let R, S be the perpendicular pressures of the ends E, D. w = weight of the beam. Then (Cor. 2. Prop. XXXI.) R $= \frac{\text{DG}}{\text{ED}} w$, and S $= \frac{\text{EG}}{\text{ED}} w$. And (Cor. 2. Prop. VIII.) S.AED : R or $\frac{\text{DG}}{\text{ED}} w$: : S.AEF : tension of DE $= \frac{\text{S.AEF} \times \text{DG}}{\text{S.AED} \times \text{ED}} w$. And S.CDE : S or $\frac{\text{EG}}{\text{ED}} w$: : S.CDH : contrary tension of DE $= \frac{\text{S.CDH} \times \text{EG}}{\text{S.CDE} \times \text{ED}} w$, and these two forces of DE being equal, we have $\frac{\text{S.AEF} \times \text{DG}}{\text{S.AED} \times \text{ED}} w =$

Fig. 21. $\frac{S.CDH \times EG}{S.CDE \times ED} w$, and $\frac{S.AEF \times DG}{S.AED} = \frac{S.CDH \times EG}{S.CDE}$.

Whence EG : DG : : $\frac{S.AEF}{S.AED} : \frac{S.CDH}{S.CDE}$: : S.AEF $\times$ S.CDE : S.CDH $\times$ S.AED.

3 *Way*.

S.CDE : S.EDS : : S : P $= \frac{S.EDS}{S.CDE} \times S$, and S.AED : S.RED : : R : $\frac{S.RED}{S.AED} R = Q$. Then P : Q : : $\frac{S.EDS}{S.CDE} \times S : \frac{S.RED}{S.AED} \times R$: : S.AED $\times$ S : S.CDE $\times$ R : : (laſt method) S.AED $\times \frac{EG}{ED} w$: S.CDE $\times \frac{DG}{ED} w$: : S.AED $\times$ EG : S.CDE $\times$ DG. And S.AED : S.CDE : : P $\times$ DG : Q $\times$ EG.

4 *Way*.

Draw C*m*, F*n* parallel to DE, and FE, HD perp. to the horizon. Then by the reſolution of forces, CD : D*m* : : P : $\frac{Dm}{CD} P$ = perpendicular force at D; and *n*E : EF : : Q : $\frac{EF}{nE} Q$ = perpendicular force at E. Therefore EG : GD : : $\frac{Dm}{CD} P : \frac{EF}{nE} Q$: : $\frac{S.CDE}{S.CmD} \times P : \frac{S.FnE}{S.nFE} \times Q$: : S.CDE $\times$ P : S.AED $\times$ Q. For S.C*m*D = S.*m*DE = S.FED = S.*n*FE. That is, EG : GD : : S.CDE $\times$ P : S.AED $\times$ Q. As in the third way.

5 *Way*.

Let R, S be the perpendicular weights of the ends E, D; or which is the ſame, the tenſions of the perpendicular ropes, FE, HD. By the resolution of forces, if Cm, Fn be parallel to DE. The force FE or R is equivalent to En, Fn; and the force Dm or S, to DC, mC; therefore EF : Fn : : R : $\frac{Fn}{FE}$R = force at E in direction Fn. And Dm : mC : : S : $\frac{mC}{mD}$S = force at D in direction mC. But the beam being in equilibrio, these two opposite forces must be equal; therefore $\frac{Fn}{FE}R = \frac{mC}{mD}S$. Whence R : S : : $\frac{mC}{mD} : \frac{Fn}{FE} : : \frac{S.CDH}{S.CDE} : \frac{S.AEF}{S.AED}$: : S.AED × S.CDH : S.AEF × S.CDE. But (Cor. 2. Prop. XXXI.) R : S : : DG : EG. Whence DG : EG : : S.AED × S.CDH : S.AEF × S.CDE; the same as by the 2d method. And the same thing likewise follows from the 1st and 4th method together. Fig. 21.

From these several ways of proceeding, it is evident, that which ever way we take, the process if rightly managed always brings us to the same conclusion; and it comes to the same thing which way we use, so that we proceed in a proper manner. And this among other things, shews the great use and extent of that noble theory of the composition and resolution of forces.

What is calculated above is concerning the angles, or the position of the several lines to one another, depending on the several forces. Then in regard to the weight of the beam, put it = w, then DC : Dm : : P : $\frac{Dm}{DC}$P = S, and En : EF : : Q : $\frac{EF}{En}$Q = R. And $w = R + S = \frac{Dm}{DC}P + \frac{EF}{En}Q$, an equation shewing the relation of the weights to one another.

 6 *Way*,

Fig. 21.

6 Way, by the center of gravity.

Produce AE, CD to B, and from B draw BGO thro' the center of gravity; which (by XXXVII.) will be perp. to AC, and therefore parallel to EF, DH. Then the angle EBG = AEF, and DBG = CDH. Then EB : BD : : S.BDE or CDE : S.BED or AED, and (Trigonom. B. II. Prop. V. Cor. 1.) EG : GD : : EB × S.EBG : BD × S.DBG : : EB × S.AEF : BD × S.CDH : : S.CDE × S.AEF : S.AED × S.CDH; the same as by the 2d way. Whence all the rest will be had as before.

Cor. *It will be exactly the same thing, whether the weights* P, Q, *remain, or the strings* AE, CD, *be fixed in that position to two tacks, any way in these lines. And if a beam* ED, *hang upon two tacks* A, C, *by ropes fixed there; it makes no difference, if you put two pulleys instead of the tacks, for the ropes to go over, and then hang on the weights* Q, P, *equal to the tensions of the strings* AE, CD. For in both cases, the forces or the tensions of the strings, and their directions, remain the same. And there is nothing else to make a difference in the situation of the beam.

SCHOLIUM.

Every one that knows any thing of mechanical principles will easily understand, that if any forces, which keep a body at rest, be resolved into others, to have the same effect; the contrary forces, or those directly opposite, must act against a single point; or else the equilibrium will be destroyed. And therefore in the present Prop. suppose any one should divide the forces CD, AE, into the two HD, DY, and FE, EX, one perpendicular, the other parallel to the horizon. The forces HD, EF, will indeed balance the force of gravity at D and E, to which

which they are directly opposite. And therefore Fig.
the beam will remain unmoved by these. But the 21.
equal forces DY, EX, being parallel, never meet in a point; but acting obliquely on the beam, one of them drawing at D in direction DY, and the other at E in direction EX, the effect will be, that they will turn the beam ED about the center of gravity G. Therefore to prevent this, the forces DY, EX, must be subdivided; that is, they must be resolved into others, one whereof is perp. to the horizon, the other parallel to ED. Then gravity will balance these perp. to the horizon, and the others, being equal and opposite, acting in the line EGD, act equally against any of the points D, G, or E. And so the beam will remain at rest. But this is much better done at once at the first, by dividing DC, EF, each into two forces, one perp. to the horizon, the other parallel to the beam ED; And then the opposite forces will exactly balance one another, and the beam remain unmoved.

PROP. XL. *Prob.*

To find the position of the beam ED, *hanging by* 22.
the rope EBD, *whose ends are fastened at* E *and* D, *and goes over a pulley fixed at* B.

Let G be the center of gravity of the beam, then (Prop. XXXVII.) BG will be perp. to the horizon. Then as the cord runs freely about the pulley B; therefore (Ax. 13.) the tension of the parts of the rope EB, BD are equal to one another, suppose $= T$. By the resolution of forces, the force EB is equivalent to EG, GB; and DB to DG, GB. Therefore BE : EG : : T : $\frac{EG}{EB}T =$ force in direction EG. And BD : DG : : T : $\frac{DG}{BD}T =$ force in

 direction

Fig. 22. direction DG, which is equal and opposite to that in EG; therefore $\frac{EG}{EB}T = \frac{DG}{DB}T$. Whence EG : EB : : DG : DB. And therefore BG bisects the angle EBD.

Cor. *Hence* ED : *string* EBD : : EG : EB *the part* EB *of the string* : : *and so* GD : DB *the part* DB *of the string.*

S C H O L I U M.

If GD be less than GE, then the center of gravity G, will be lowest, when the beam hangs perpendicular with the end D downward. And in many cases it will be highest, when it hangs perpendicular, with the end E downward.

P R O P. XLI. *Prob.*

23. *There is a beam* BC *hanging by a pin at* C, *and lying upon the wall* BE; *to find the forces or pressures at the points* B, *and* C, *and their directions.*

Produce BC to K, so that CK may be equal to CB. Draw BA parallel, and CL perpendicular, to the horizon; and draw BL, CN, KI perp. to BCK. Thro' the center of gravity G, draw GF parallel to CL. By Prop. XIV. if a body lies upon an inclined plane, as BC; its weight, its inclination down the plane, and pressure against it, are as BC, CA and AB; that is, as CL, CB and BL. Therefore if CL represent the weight of the body, CB will be the force urging it down the plane, and BL the total pressure against the plane. And since GF is parallel to CL, BL is divided in F, in the same ratio, as BC is divided in G. And therefore (Cor. 2. Prop. XXXI.) BF will be the part of the pressure acting at C, and FL the part acting at B. Make CN equal to BF, and compleat the

the parallelogram CNIK, and draw CI. Then since BC or CK is the force in direction CK, and CN the force in direction CN; then by composition, CI will be the single force by which C is sustained, and CI its direction. But the triangles CKI, CBF are similar and equal, and CI = CF, and in the same right line; therefore CF is the quantity and direction of the force acting at C to sustain it. Therefore the weight of the body, the pressure at B, and the force at C; are respectively as CL, FL, and CF. Fig. 23.

Cor. 1. *Produce* FG *to intersect* CN *in* H; *then the weight of the body, the pressure at* B, *and the force acting at* C; *are respectively as* HF, HC, *and* CF.

For in the parallelogram CLFH, HF = CL, and HC = FL.

Cor. 2. *If the beam was supported by a pin at* B, *and laid upon the wall* AC; *the like construction must be made at* B, *as has been done at* C, *and then the forces and directions will be had.*

Cor. 3. *If a line perp. to the horizon be drawn thro'* F, *where the direction of the forces* CF, *and* BF *meet; it passes thro'* G *the center of gravity of the beam.*

Cor. 4. *It is all one whether the beam is sustained by the pin* C *and the wall* BE, *or by two ropes* CI, BP *acting in the directions* FC, FB, *and with the forces* CF, FL.

SCHOLIUM.

The proportions and directions of the forces here found, are the same as in Prop. LXIV. of my large book of Mechanics, and obtained here by a different method. The principles here used are indisputable; and the principle made use of in that

 LXIV.

Fig. LXIV. Prop. is here demonſtrated in the third Cor.
23. ſo that the reader may depend upon the truth of them all.

PROP. XLII. *Prob.*

24. BC *is a heavy beam ſupported upon two poſts* KB, LC; *to find the poſition of the poſts, that the beam may reſt in equilibrio.*

Let G be the center of gravity; draw BA parallel to the horizon, and BF, GD, CAN perpendicular to it. Then (Prop. XXXI. Cor. 2.) if BC be the weight of the body, CG will be the part of the weight acting at B, and BG the weight at C. Therefore make CN = BG, and BF = GC; and from N and F, draw NI, FK, parallel to BC; and make NI = FK, of any length, and lying contrary ways. Then draw IC and KB, which will be the poſition of the poſts required.

For BF is the weight upon B; and CN, that upon C, which forces being in direction of the lines BF, CN, the beam will remain at reſt by theſe forces. And the forces NI, FK, in direction BC, being equal and contrary, will alſo keep the beam in equilibrio, therefore the forces KB, IC, compounded of the others, will alſo keep the beam in equilibrio, acting in the directions KB, IC, or MB, LC.

Cor. 1. *Hence if* DG *be produced, it will paſs thro' the interſection* H, *of the lines* LC, MB.

For the triangles INC, CGH are ſimilar; therefore IN : NC : : CG : GH, the interſection with CL. Alſo the triangles KFB, BGH are ſimilar; therefore KF : BG : : BF : GH the interſection with MB, which muſt needs be the ſame as the other, ſince the three firſt terms of the proportion are the ſame; for KF = NI, BG = NC, and BF = CG.

Cor.

Cor. 2. *If a line be drawn thro' the center of gravity* G, *of the beam, perpendicular to the horizon; and from any point* H *in that line, (above or below* G), *the lines* HBK *and* HCM *be drawn; then the props* BM *and* CL *will sustain the beam in equilibrio.* Fig. 24.

Cor. 3. *If* GO *be drawn parallel to* HC; *then the weight of the beam, the pressure at* C, *and thrust or pressure at* B; *are respectively as* HG, OG, *and* HO, *and in these directions.*

Cor. 4. *It is all one for maintaining the equilibrium, whether the beam* BC *be supported by two posts or props* MB, LC; *or by two ropes* BH, CH; *or by two planes perpendicular to these ropes at* B *and* C.

For in all these cases the forces and directions are the same; and there is nothing else concerned, but the forces and directions.

SCHOLIUM.

It does not always happen that the center of gravity is at the lowest place it can get, to make an equilibrium. For here if the beam BC be supported by the posts MB, LC; the center of gravity is at the highest it can get; and being in that position, it has no inclination to move one way more than another, and therefore it is as truly in equilibrio, as if it was at the lowest point. It is true, any the least force will destroy that equilibrium, and then if the beam and posts be movable about the angles M, B, C, L, which is all along supposed, the beam will descend till it is below the points M, L, and gain such a position as described in Prop. XXXIX. and its Cor. supposing the ropes fixed at A, C (fig. 21.); and then G will be at the lowest point, and will come to an equilibrium again. In planes, the center of gravity G may be either at its highest or lowest point. And there are cases, when the

Fig. 24. the center of gravity is neither at its highest nor lowest, as may happen in the case of Prop. XL. so that they who contend, that in case of an equilibrium, the center of gravity must *always* be at the lowest place, are much mistaken, and know little about the principles of mechanics, or the nature of things.

Those that want to see more variety about the motion of bodies, on inclined planes; the pressure, and direction of the pressure of beams of timber; centers of gravity, and also the centers of oscillation and percussion, &c. may consult my large book of Mechanics.

SECT.

Fig.

SECT. IV.

The MECHANICAL POWERS; *the Strength and Streſs of Timber.*

PROP. XLIII.

IN a balance, where the arms are of equal length; if two equal weights be ſuſpended at the ends, they will be in equilibrio.

The balance is a ſtreight inflexible rod or beam, turning about a fixed point or axle in the middle of it; to be loaded at each end with weights ſuſpended there.

Let AB be the beam or lever, C the middle 25
point or center of motion; D, E the weights hanging at the ends A and B. Then let the beam and the weights, or the whole machine, be ſuſpended at C. And ſuppoſe the beam and the weights be turned about upon the center C; then the points A, B being equidiſtant from C will deſcribe equal arches, and therefore the velocities will be equal, and if the bodies D and E be equal, then the motion of D will be equal to the motion of E, as their quantities of matter and velocities are equal; and conſequently, if the beam and weights are ſet at reſt, neither of them can move the other, but they will remain in equilibrio.

Cor. *If one weight be greater than the other; that weight and ſcale will deſcend, and raiſe the other.*

SCHOLIUM.

The uſe of the balance, or a common pair of ſcales, is to compare the weights of different bodies;

Fig. dies; for any body whofe weight is required, be-
25. ing put into one fcale, and balanced by known weights put into the other fcale, thefe weights will fhew the weight of the body. To have a pair of fcales perfect, they muft have thefe properties. 1. The points of fufpenfion of the fcales, and the center of motion of the beam, A, C, B, muft be in a right line. 2. The arms AC, BC, muft be of equal length from the center. 3. That the center of gravity be in the center of motion C. 4. That there be as little friction as poffible. 5. That they be in equilibrio, when empty.

If the center of gravity of the beam be above the center of motion, and the fcales be in equilibrio, if they be put a little out of that pofition, by putting down one end of the beam, that end will continually defcend, till the motion of the beam is ftopt by the handle H. For by that motion, the center of gravity is continually defcending, according to the nature of it. But if the center of gravity of the beam be below the center of motion; if one end of the beam be put down a little, to deftroy the equilibrium; it will return back and vibrate up and down. For by this motion the center of gravity is endeavouring to defcend.

PROP. XLIV. *Prob.*

25. *To make a falfe balance; or one which is in equilibrio when empty, and alfo in equilibrio, when loaded with unequal weights.*

Make fuch a balance as defcribed in the laft Prop. only inftead of making the center of motion in the middle at C, make it nearer one end, as at F. And make the weight of the fcales fuch, that they may be in equilibrio upon the center F. Then if two weights D, E, be made to be in equilibrio in the two fcales; thefe weights will be unequal, for

for they will be reciprocally as the lengths of the arms AF, BF. That is, AF : BF : : E : D. Fig. 25.

For (Prop. XXXI. Cor. 1.) since F is the center of gravity of D and E, supposing them to act at A and B; therefore $FA \times D = FB \times E$. And FA : FB : : E : D. But AF is greater than FB, therefore E is greater than D.

Cor. 1. *Hence to discover a false balance, make the weights in the two scales to be in equilibrio; then change the weights to the contrary scales. And if they be not in equilibrio, the balance is false.*

Cor. 2. *Hence also to prove a pair of good scales, they must be in equilibrio when empty, and likewise in equilibrio with two weights. Then if the weights be changed to the contrary scales, the equilibrium will still remain, if the scales are good.*

Cor. 3. *From hence also may be known what is gained or lost, by changing the weights, in a false balance.*

Take any weight as 1 pound, to be put into one scale and balanced by any sort of goods in the other. Since $AF \times D = BF \times E$; let the weight D be 1, then $E = \frac{AF}{BF}$ the weight of the goods in the scale E. Then changing the scales, let the weight E be 1; then $D = \frac{BF}{AF}$ the weight of the goods in the scale D. Then $\frac{AF}{BF} + \frac{BF}{AF} =$ whole weight of the goods, and $\frac{AF}{BF} + \frac{BF}{AF} - 2 =$ gain to the buyer in 2 lb. &c. Therefore

$$\frac{AF^2 + BF^2 - 2AF \times BF}{AF \times BF} = \text{gain in 2 lb.} = \frac{\overline{AF - BF}|^2}{AF \times BF}; \text{ and } \frac{\overline{AF - BF}|^2}{2AF \times BF} = \text{gain in 1 lb.}$$

There-

Fig. Therefore if w is any weight to be bought; the
25. gain to the buyer, for the weight w, by changing the ſcales, will be $\frac{\overline{AF-BF}|^2}{2AF \times BF}w$. For example, if AF be 16, and BF 15; then the gain will be $\frac{\overline{16-15}|^2}{2 \times 16 \times 15}w = \frac{1}{480}w$.

Scholium.

In demonſtrating the properties of the mechanical powers; ſince the weight is commonly ſome large body whoſe weight is to be overcome or balanced; therefore the power which acts againſt it, will be moſt fitly repreſented by another weight; and thus the power and weight being of the ſame kind, may moſt properly and naturally be compared together. For ſuch a weight may repreſent any power, as the ſtrength of a man's hand, the force of water or wind, &c. And this weight repreſenting the power, being ſuſpended by a rope, may hang perpendicular where the power is to act perpendicular to the horizon; or may go over a pulley, when it acts obliquely.

PROP. XLV.

6. *If the power and weight be in equilibrio upon any*
7. *lever, and act in lines perpendicular to the lever;*
8. *then the power* P *is to the weight* W; *as the diſ-*
9. *tance of the weight from the ſupport* C, *is to the diſtance of the power from the ſupport.*

There are four ſorts of levers. 1. When the ſupport is between the weight and the power. 2. When the weight is between the power and the ſupport. 3. When the power is between the weight and the ſupport. 4. When the lever is not ſtreight but crooked.

A le-

A lever is any inflexible rod or beam, of wood Fig.
or metal, made use of to move bodies. The sup- 26.
port is the prop it rests on, in moving or sustaining 27.
any heavy body, and this is the same as the center 28.
of motion. 29.

Let the power P act at A by means of a rope; then as C is the prop or center of motion, if the lever be made to move about the center C, the arches described by A and W; that is, the velocities of A and W will be as the radii CA and CW. But the velocity of P is the same as that of the point A. Therefore velocity of P : velocity of W : : CA : CW : : (by supposition) W : P; therefore P × velocity of P = W × velocity of W. Consequently their motions are equal, and they cannot move one another, but must remain in equilibrio. And if they be in equilibrio, they must have this proportion assigned.

Cor. 1. *If a power* P *act obliquely against the le-* 30.
ver WA; *the power and weight will be in equilibrio, when the power* P *is to the weight* W; *as the distance of the weight* CW, *to* CD *the perpendicular, drawn from the support to the line of direction of the power.*

For in this case WCD becomes a bended lever, and the power P acts perpendicular to CD at D; and (Ax. 12.) it is all one whether the power acts at D or A, in the line of direction AD. And hence,

Cor. 2. *If any force be applied to a lever* ACW *at* A, *its power to turn it about the center of motion* C, *is as the sine of the angle of application* CAD.

For if CA be given, CD is as the sine of CAD.

Cor. 3. *In a streight lever, of these three, the power, the weight, and the pressure upon the support; the middlemost is equal to the sum of the other two.*

For the middle one acts against both the others and supports them.

Cor.

Fig. Cor. 4. *From the foregoing properties of the le-*
30. *ver, the effects of several sorts of simple machines may be explained; and likewise the manner of lifting, carrying, drawing of heavy bodies, and such like.*

26. For example, if a given weight W is to be raised by a small power applied at A, the end of the lever AW. If we divide WA in C, so that it be as CA : CW : : as the weight W : to the power P; then fixing a prop or support at C or rather a little nearer W; then the power P with a small addition, will raise the weight W.

27. Again, if two men be to carry a weight W, upon the lever CA. The weight the man at A carries, is to the weight the man at C carries as CW, to AW. And therefore the lever or beam CA ought to be divided in that proportion at W, the place where the weight is to lie, according to the strength of the men that carry it.

31. Likewise if two horses be to draw at the swing-tree AB, by the ropes AF, BG; and the swing-tree draws a carriage &c. by the rope CD; then the force acting at A will be to the force acting at B, as BC to AC. And therefore BC ought to be to AC, as the strength of the horse at F, to the strength of the horse at G; the weaker horse having the longer end. But it is proper to make the cross bar AB crooked at C; that when the stronger horse pulls his end more forward, he may pull obliquely, and at a less distance from the center; whilst the weaker horse pulls at right angles to his end, and at a greater distance.

Again, such instruments as crows and handspikes are levers of the first kind, and are very useful and handy for raising a great weight to a small hight. Also sciffars, pinchers, snuffers, are double levers of the first kind, where the joint is the fulcrum or support. The oars of a boat, the rudder of a ship, cutting knives fixed at one end, are levers of the

second

second kind. Tongs, sheers, and the bones of Fig.
animals, are levers of the third kind, a ladder raised upright, is a lever of the third kind. A hammer drawing out a nail is a lever of the fourth kind. 32.

The *Steel Yard* is nothing but a lever of the first kind, whose mechanism or construction is this. Let AB be the beam, C the point of suspension; P the power, movable along the beam CB. The beam being suspended at D, move the power P, along towards C, till you find the point O, where P reduces the beam to an equilibrium. Then at A hang on the weight W of 1 pound; and move P to be in equilibrio with it at 1; then hang on W of 2 pound, and shift P till it be in equilibrio, at 2. Proceed thus with 3, 4, 5, &c. pounds at W, and find the divisions 3, 4, 5, &c. Or if you will; after having found the points O, 1; make the divisions, 12, 23, 34, &c. each equal to O1. But for more exactness and expedition, having found the point O, where P makes the beam in equilibrio: hang on any known number of pounds, as W; and move P to the point B, where it makes an equilibrium. Then divide OB into as many equal parts as W consists of pounds: mark these divisions 1, 2, 3, 4, &c. Then any weight W being suspended at A. If P be placed to make an equilibrium therewith; then the number where P hangs will shew the pounds or weight of W.

To prove this, we must observe, that AC is the heavier end of the beam; therefore let Q be the *Momentum* at that end to make an equilibrium with P suspended at O; that is, let $Q = CO \times P$. But (Cor. 2. Prop. XXXVI.) $Q + CA \times W = CF \times P = CO \times P + OF \times P$. Take away $Q = CO \times P$, and then $CA \times W = OF \times P$. Whence $AC : P :: OF : W$. But AC and P are always the same; therefore W is as OF; that is,

F if

Fig. if OF be 1, 2, 3, &c. divisions, then W is 1, 2, 3,
32. &c. pounds.

We may take notice that the divisions O1, 12, 23, &c. are all equal; but CO may be greater or lesser, or nothing.

If you would know how much the weight P is, take the distance CA, and set it from O along the divisions O, 1, 2, 3, &c. and it will reach to the number of pounds. But this is of no consequence, being only matter of curiosity.

PROP. XLVI.

33. *In the compound lever, or where several levers act upon one another, as* AB, BC, CD, *whose supports are* F, G, I; *the power* P : *is to the weight* W : : *as* BF × CG × DI : *to* AF × BG × CI.

For the power P acting at A : force at B : : BF : AF; and force or power at B : force at C : : CG : GB; and force or power at C : weight W : : DI : IC. Therefore *ex equo*, power P : weight W : : BF × CG × DI : AF × GB × IC.

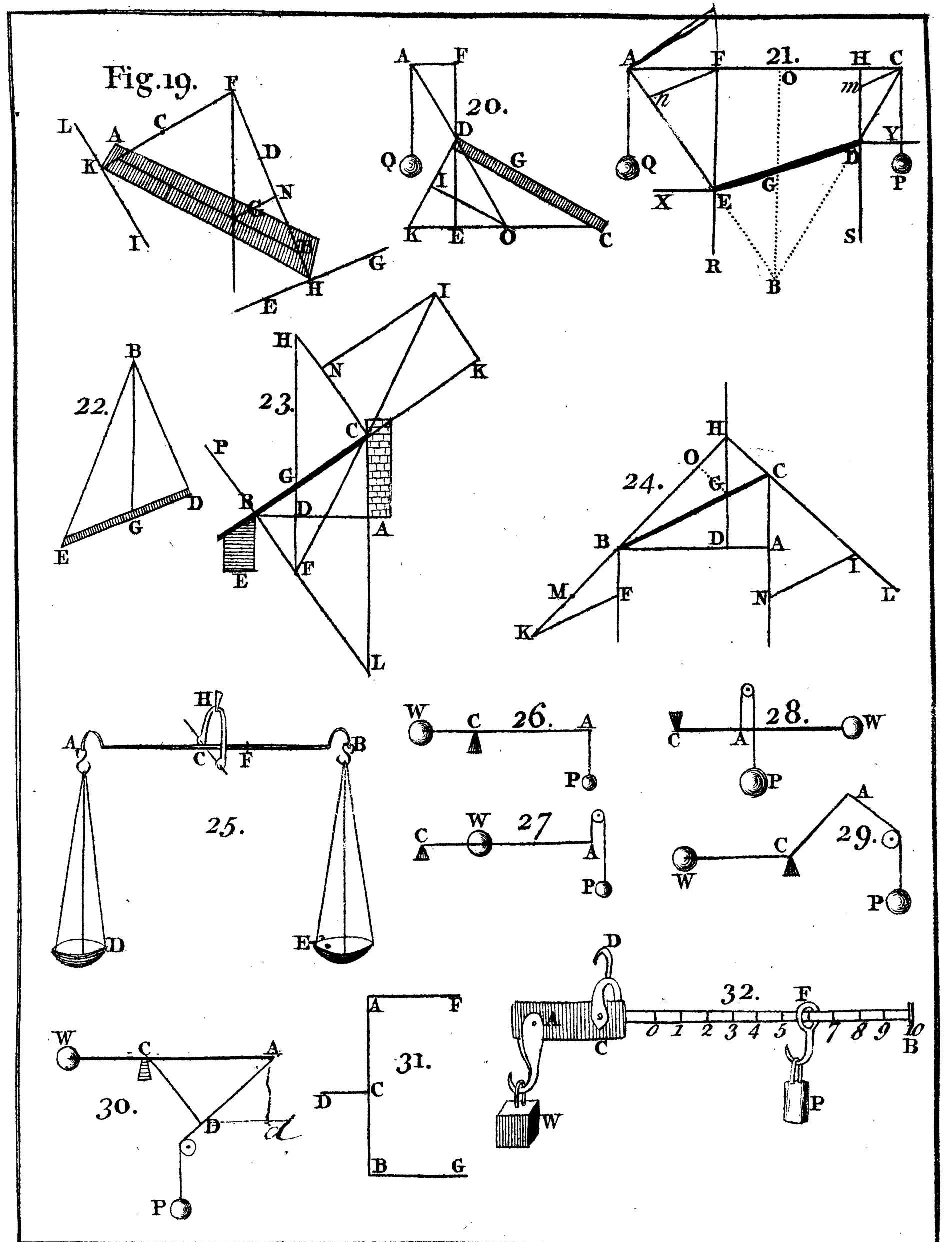
And it is the same thing in the other kinds of levers, taking the respective distances, from the several props or supports.

PROP. XLVII.

34. *In the wheel and axle; the weight and power will be in equilibrio, when the power* P *is to the weight* W; *as the radius of the axle* CA, *where the weight hangs; to the radius of the wheel* CB, *where the power acts.*

This is a wheel fixed to a cylindrical roller, turning round upon a small axis; and having a rope going round it.

Thro'



Mechanics

Pl. II. pa. 76.

Thro' the center of the wheel C, draw the horizontal line BCA. Then BP and AW are perpendicular to BA; and BCA will be a lever whose support is C; and the power acts always at the distance BC, and the weight at the distance CA; which remain always the same. Therefore the weight and power act always upon the lever BCA. But by the property of the lever (Prop. XLV.) BC : CA : : W : P, to have an equilibrium. Fig. 34.

Otherwise,

If the wheel be set a moving the velocity of the point A or of W, is to that of B or P, as CA to CB; that is (by supposition), as P : W. Therefore W × velocity of W = P × velocity of P; therefore the motions of P and W, being equal, they cannot, when at rest, move one another.

Cor. 1. *If the power acting at the radius* CB, *act not at right angles to it; draw* CD *perpendicular to* BP *the direction of the power; then the power* P : *is to the weight* W : : *as the radius of the axle* CA : *to the perpendicular* CD. 35.

For in the lever DCA, whose support is C, the power P : weight W : : CA : CD.

Cor. 2. *In a roller turned round, on the axis or spindle* FC, *by the handle* CBG; *the power applied perpendicularly to* BC *at* B, *is to the weight* W : : *as the radius of the roller* DA, *to the length of the handle* CB. 36.

For in turning round, the point B describes the circumference of a circle; the same as if it was a wheel whose radius is CB.

SCHOLIUM.

All this is upon supposition that the rope sustaining the weight is of no sensible thickness. But if it is a thick rope, or if there be several folds of

Fig. it about the roller or barrel; you muſt meaſure
36. to the middle of the out ſide rope to get the radius of the roller. For the diſtance of the weight from the center is increaſed ſo much, by the rope's going round.

From hence the effects of ſeveral ſorts of machines, or inſtruments, may be accounted for. A roller and handle for a well or a mine, is the ſame thing as a wheel and axle, a windleſs and a capſtain in a ſhip is the ſame; and ſo is a crane to draw up goods with. A gimblet and an auger to bore with, may be referred to the wheel and axle.

The wheel and axle has a particular advantage over the lever; for a weight can but be raiſed a very little way by the lever. But by continual turning round of the wheel and roller, the weight may be raiſed to any hight required.

PROP. XLVIII.

37. *In a combination of wheels with teeth; if the power* P *be to the weight* W : : *as the product of the diameters of all the axles or pinions, to the product of the diameters of all the wheels; the power and weight will be in equilibrio.*

AC, CD are the radii of one wheel and its axle; DG, GH, the radii of another; and HI, IK are thoſe of another. Theſe act upon one another at D and H, then as the power or force P is propagated thro' all the wheels and axles to W; we muſt proceed to find the ſeveral forces acting upon them, by Prop. XXXVII. Thus,

$$CD : CA : : P : \frac{CA}{CD} P = \text{force acting at D.}$$

$$\text{and } GH : GD : : \frac{CA}{CD} P \text{ (force at D)} : \frac{CA \times GD}{CD \times GH} P$$
$$= \text{force acting at H.}$$

and

and IK : IH : : $\frac{CA \times GD}{CD \times GH}$ P (force at H) : Fig 37.
$\frac{CA \times GD \times IH}{CD \times GH \times IK}$ P = force at K = W. And CA $\times$ GD $\times$ IH $\times$ P = CD $\times$ GH $\times$ IK $\times$ W; whence P : W : : CD $\times$ GH $\times$ IK : : CA $\times$ GD $\times$ IH.

Cor. 1. *If the weight and power be in equilibrio, and made to move; the velocity of the weight, is to the velocity of the power; as the product of the diameters of all the axles or pinions, to the product of the diameters of all the wheels. Or instead of the diameters, take the number of teeth in these axles and wheels that drive one another. And the same is true of wheels carried about by ropes.*

For the power is to the weight; as the velocity of the weight to the velocity of the power. And the number of teeth in the wheels and pinions, that drive one another, are as the diameters. And the ropes supply the place of teeth.

Cor. 2. *In a combination of wheels with teeth. The number of revolutions of the first wheel, is to the number of revolutions of the last wheel, in any time; as the product of the diameters of the pinions or axles, to the product of the diameters of the wheels: or as the product of the number of teeth in the pinions, to the product of the number of teeth in the wheels which drive them. And the same is true of wheels going by cords.*

For as often as the number of teeth in any pinion, is contained in the number of teeth of the wheel that drives it; so many revolutions does that pinion make for one revolution of the wheel.

SCHOLIUM.

A pinion is nothing but a small wheel, fixed at the other end of the axis, opposite to the wheel; and

Fig. 37. and consists but of a few leaves or teeth; and therefore is commonly less than the wheel. But in the sense of this proposition, a pinion may, if we please, be bigger than the wheel. As if we put the power and weight into the contrary places, the wheels will become the pinions, and the pinions the wheels, according to the meaning of this proposition.

PROP. XLIX.

If a power sustains a weight by means of a fixed pulley; the power and weight are equal: but if the pulley be movable along with the weight, then the weight is double the power.

A pulley is a small wheel of wood or metal, turning round upon an axis, fixed in a block; on the edge of the pulley is a groove for the rope to go over.

38. 39. Thro' the centers of the pullies, draw the horizontal lines AB, CD; then will AB represent a lever of the first kind, and its support is the center of the pulley, which is a fixed point, the block being fixed at F. And the points A, and B, where the power and weight act, being equally distant from the support, therefore (Prop. XLV.) the power P = weight W.

Also CD represents a lever of the second kind, whose support is at C, a fixed point; the rope CG being fixed at G. And the weight W acting at the middle of CD, and the power acting at D, twice the distance from C; therefore (Prop. XLV.) the power P is to the weight W : : as $\frac{1}{2}$ CD to CD; or as 1 to 2.

Cor. *Hence all fixed pulleys are levers of the first kind, and serve only to change the direction of the motion; but make no addition at all to the power. And*

And therefore if a rope goes over several fixed pullies; the power is not increased, but rather decreased, by the friction. Fig. 38. 39.

SCHOLIUM.

The use of a fixed pulley is of great service in raising a weight to any height, which otherwise must be carried by strength of men, which is often impracticable. Therefore if a rope is fixed to the weight at W (fig. 38.) and passed over the pulley BA; a man taking hold at P will draw up the weight, without moving from the place. And if the weight be large, several persons may pull together at F, to raise the weight up; where in many cases they cannot come to it, to raise it by strength.

PROP. L.

In a combination of pullies, all drawn by one rope going over all the pullies; if the power P *is to the weight* W; *as* 1 *to the number of the parts of the rope proceeding from the movable block and pullies. Then the power and weight will be in equilibrio.* 40.

Let the rope go from the power about the pullies in this order, *ntovrs*, where the last part *s* is fixed to the lower block B. Now (Ax. 13.) all the parts of the rope *ntovrs* are equally stretched, and therefore each of them bears an equal weight; but the part *n* bears the power P, which goes to the fixed block A. All the other parts, sustain the weight and movable block B, each with a force equal to P. Therefore P is to the sum of all the forces, sustained by *o*, *r*, *s*, *t*, *v*, or the weight W, as 1 to the number of these ropes immediately communicating with the movable block B. And all the ropes having an equal tension, none of them

Fig. 40. them can move the reſt, but they muſt remain in equilibrio.

And if you take away the power at P, and apply a force at the rope *t* equal to P, to pull upwards in direction *t*A; this will make no alteration, for the rope *t* draws from the movable block with the ſame force as before, and therefore the weight is ſuſtained as before; for the upper pulley (by Prop. XLIX. Cor.) which the rope *nt* goes over, ſerves only to change the direction. And therefore as there are the ſame number of ropes ſtill drawing from the movable block as before; the propoſition holds good alſo in this reſpect. And it would be the ſame thing if the rope *s* was fixed to the weight W inſtead of the block B; but had it been fixed to the block A, there muſt have been a pulley more below, and a rope more, which would have increaſed the power, according to the propoſition.

Cor. 1. *Hence it appears to be a diſadvantage to the power to pull againſt the fixed block.*

For the rope *n* has no more purchaſe, or no more effect than the rope *t* has which draws againſt the movable block; and therefore when one draws by the rope *n*, there muſt be a pulley more, which will create more friction.

Cor. 2. *Hence one may explain the effects of all ſorts of machines compoſed of pullies; or find out ſuch a conſtruction or combination of them as to anſwer any purpoſe deſired. And to find its force, begin at the power, and call it* 1; *then all parts of the running rope that go and return about ſeveral pullies, muſt be each numbered alike. And any rope that acts againſt ſeveral others muſt be numbered with the ſum of theſe. And ſo on to the weight.*

For

For example; ſuppoſe a man wanted to draw himſelf up to the top of a houſe or a church. Get a pulley A fixed at the top, and place another B at the bottom. Let a rope be fixed to the upper block A, and brought down about the pulley B, and then put round the upper pulley, and ſo brought to the ground at H. Then if a croſs ſtick CD be faſtened to the block B by a rope; a man may get aſtride of the ſtick, and then draw himſelf up by the rope H. And the power to draw himſelf up, will be little more than $\frac{1}{3}$ of his weight. For the power at H, and the two parts of the rope going about the pulley B, ſuſtain all his weight; and each of them ſuſtains one third of it. Fig. 41.

If inſtead of the ſtick CD, he takes a chair to ſit in; then when he has drawn himſelf up to any hight he pleaſes, he may fix the rope H to the chair, and then do any ſort of buſineſs, as ſet up a dial, point the walls, and ſuch like, as is commonly done.

Again, ſeveral tackles are uſed aboard a ſhip, for hoiſting goods and the like. Let A, B be two blocks with pullies, the upper one being fixed, and let a weight W be ſuſpended at the ſingle pulley and rope, one end of the rope being fixed at F, and the other faſtened to the movable block B. This pulley and rope BCF is called a *Runner*. Let the power be at P, call it 1; then all the ropes going from B to A, muſt be each of them 1, and the rope going from the block B, acting againſt theſe four muſt be marked 4, and the other part of it CF muſt alſo be 4. Laſtly, the weight acting againſt theſe two, muſt be 8. And then the power P is to the weight W, as 1 to 8. 42.

ABCD is another tackle with a runner BAD, A being a fixed pulley; the two blocks B, C, are both movable. The rope DAB is fixed to the weight 43.

Fig. 43. weight at D, and to the block B. The rope PB goes and returns about the pullies BC, and at last is fastened to the block C. Let P be the power, mark it 1, then the other parts of the rope between the blocks, must also be 1 apiece. Then CI acting against 3, must be 3. And AB is 4, as it acts against 4; likewise AD must be 4. Therefore the whole force that sustains the weight W is 3 and 4, or 7. And the power to the weight as 1 to 7.

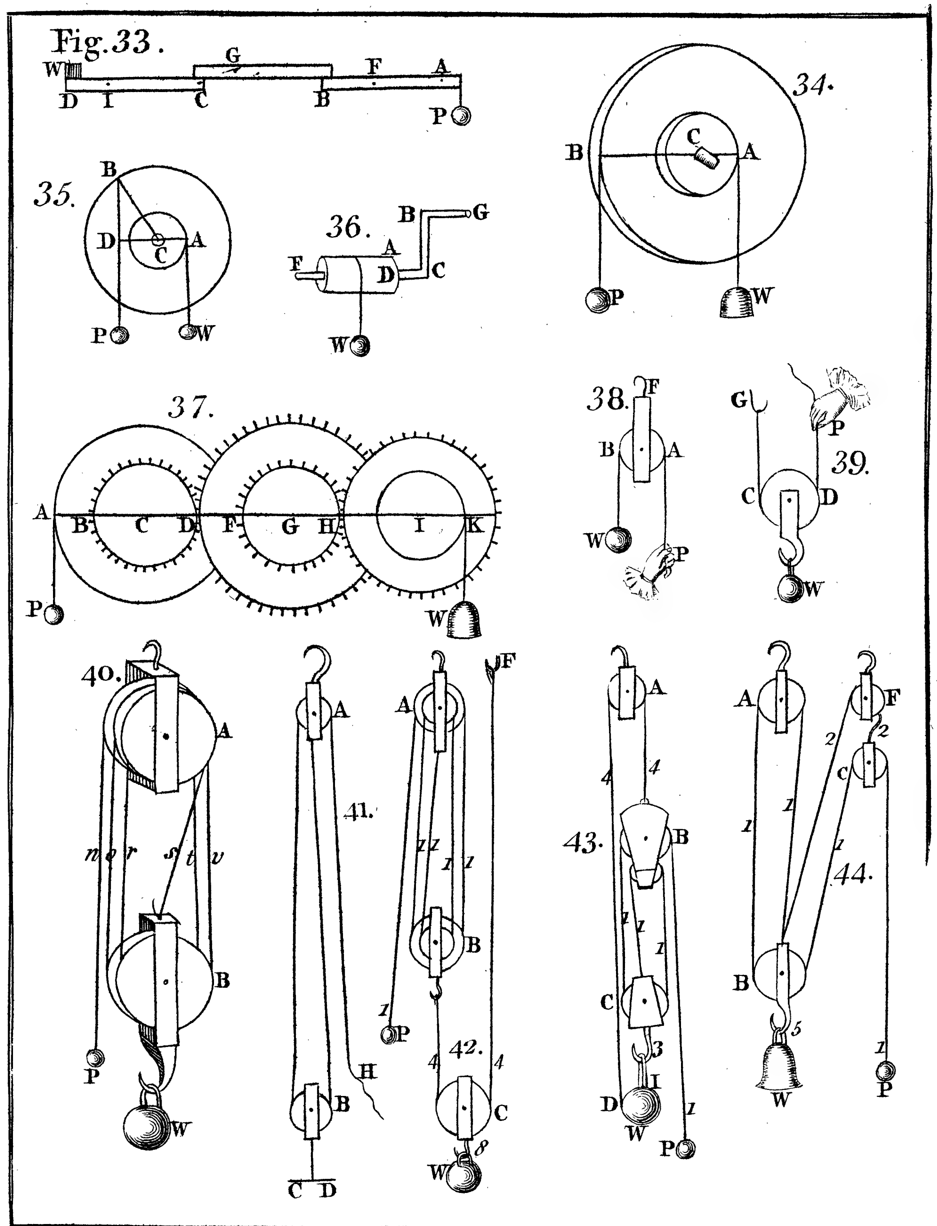
44. The following is a sort of *Spanish burton*, A and F are two fixed pullies; C and B two movable ones. The rope going from the power P, goes round C, B, and A, and is fastened to the block B. Another rope is fastened to the block B, and goes over the pulley F, and is fixed to the block C. Then marking the power P, 1. Then each part of the rope, continued over C, B, and A to B again must be each 1. Then FC must be 2, as it acts against two parts; and likewise the other part of it FB must be 2. Then the whole that lifts the weight W, is $1 + 1 + 1 + 2 = 5$. And therefore the power is to the weight as 1 to 5.

The friction between the pullies and blocks is sometimes considerable. To remedy which, they must be as large as they can conveniently be made, and kept oiled or greased.

PROP. LI.

45. *In the screw, if the power applied at* E, *be to the weight, pressure, &c. at* B; *as the distance of two threads of the screw, taken parallel to the axis of it, is to the circumference described by the power at* E; *then the weight and power will be in equilibrio.*

A screw is an instrument consisting of two parts AB, CD, fitting into one another. AB is the male screw, called the top or spindle; this is a long cylindrical body, having its surface cut into ridges and



Mechanics

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and hollows, that run round it in a ſpiral manner Fig.
from one end to the other, at equal diſtances; 45.
theſe riſings are called threads, and ſo many revolutions as they make, ſo many threads the ſcrew contains. CD is the female ſcrew, or the plate, thro' which the other goes; its concavity is cut in the ſame manner as the male, ſo that the ridges of the male may exactly fit the hollows of the female. By reaſon of the winding of the threads, as the handle EF is turned one way or the other, the male AB goes further in or comes further out, of the female.

Let the point E of the handle, make one revolution, then the male AB will have advanced the diſtance between one thread and another, of the ſcrew. Therefore if G repreſent any weight, which the end B acts againſt, it will be moved thro' the breadth of a thread, whilſt the power moves thro' a circumference whoſe radius is EA. Therefore the velocity of G is to the velocity of E, as the breadth of a thread, to the circumference deſcribed by E; that is (by ſuppoſition) as the power at E, to the weight at G. Therefore E $\times$ velocity of E $=$ G $\times$ velocity of G; and therefore their motions being equal, they will be in equilibrio.

Cor. *By reaſon of the friction, if any weight is to be removed by a ſcrew; the power muſt be to the weight; at leaſt as the breadth of two threads of the ſcrew, to the circumference deſcribed by the power; to keep the weight in equilibrio; and muſt be much more to move it.*

For in the ſcrew there is ſo much friction, that it will ſuſtain the weight when the power is taken away. And therefore the friction is as great or greater than the power. And therefore the whole power applied muſt at leaſt be doubled to produce any motion.

SCHO-

Fig. 45.

Scholium.

Screws with ſharp threads have far more friction than thoſe with ſquare threads, and therefore move a body with more difficulty.

The ſcrew, in moving a body, acts like an inclined plane. For it is juſt the ſame as if an inclined plane was forced under a body to raiſe it; the body being prevented from flying back, and the baſe of the plane being driven parallel to the horizon.

The uſe of this power is very great. It is of great ſervice for fixing ſeveral things together by help of ſcrew nails; it is likewiſe very uſeful for ſqueezing or preſſing things cloſe together, or breaking them; alſo for raiſing or moving large bodies. The ſcrew is uſed in preſſes for wine, oil, or for ſqueezing the juice out of any fruit. The very friction of this machine has its particular uſe, for when a weight is raiſed to any hight; if the power be taken away, the ſcrew will retain its poſition, and hinder the weight from deſcending again by its friction, without any other power to ſuſtain it.

In the common ſcrew, ſuch as is here ſuppoſed; the threads are all one continued ſpiral from one end to the other; but where there are two or more ſpirals, independent of one another, as in the worm of a jack; you muſt meaſure between thread and thread of the ſame ſpiral, in computing the power.

PROP.

Fig.

PROP. LII.

In the endless screw, where the teeth of the worm or spindle AB, *drives the wheel* CD, *by acting against the teeth of it. If the power applied at* P, *is to the weight* W, *acting upon the edge of the wheel at* C : : *as the distance of two threads or teeth, between fore side and fore side, taken along* AB ; *is to the circumference described by the power* P. *Then the weight and power will be in equilibrio.* 46.

The *endless* or *perpetual screw* is one that turns perpetually round the axis AB ; and whose teeth fit exactly into the teeth of the wheel CD, which are cut obliquely to answer them: So that as AB turns round, its teeth take hold of the teeth of the wheel CD, and turns it about the axis I, and raises the weight W.

For by one revolution of the power at P, the wheel will be drawn forward one tooth ; and the weight W will be raised the same distance. Therefore the velocity of the power, will be to that of the weight ; as that circumference, to one tooth : : that is (by supposition) as the weight W, to the power P. Therefore the power P × velocity of P = W × velocity of W ; therefore their motions being equal, they will be in equilibrio.

Cor. *If a weight* N *be suspended at* E *on the axle* EF ; *then if the power* P, *is to the weight* N : : *as the breadth of a tooth* × EF, *to the circumference described by* P × CD. *They will be in equilibrio.*

Or if the power P ; *is to the weight* N : : *as radius of the axle* EI, *to the radius of the handle* FP × *by the number of teeth in* CD ; *they will be in equilibrio.*

For

Fig. For power P : weight W : : 1 tooth : circumference.
46. and weight W : weight N : : EF : CD.
therefore P : N : : 1 tooth × EF : CD × circumference.

Or thus, whilst EF turns round once, P turns round as oft as CD has teeth; whence EI : BP × number of teeth : : velocity of N : velocity of P : : P : N.

SCHOLIUM.

47. As the teeth of the wheel CD, must be cut obliquely to answer the teeth or screw on AB; supposing AB to lie in the plane of the wheel CD; and therefore the wheel will be acted on obliquely by the screw AB. To remedy that, the screw AB may be placed oblique to the wheel, in such a position, that when the teeth of the wheel are cut streight or perp. to its plane, the teeth of the screw AB, may coincide with them, and fit them. By that means the force will be directed along the plane of the wheel CD. Fig. 47 explains my meaning.

This machine is of excellent use, not only in itself, for raising great weights, and other purposes; but in the construction of several sorts of compound engines.

PROP. LIII.

48. *In the wedge* ACD, *if a power acting perpendicular to the back* CD, *is to the force acting against either side* AC, *in a direction perpendicular to it; as the back* CD, *to either of the sides* AC; *the wedge will be in equilibrio.*

A wedge is a body of iron or some hard substance in form of a prism, contained between two isosceles triangles, as CAD. AB is the hight, and CD the back of it; AC, AD the sides.

Let AB be perp. to the back CD, and BE, BF, perp. to the sides AC, AD. Draw EG, FG parallel

rallel to BF, BE; then all the sides of the parallelogram BEGF are equal. The triangles EGB, ADC are similar; for draw EOF which will be perp. to AB; then the right angled triangles AEB, AEO, are similar, and the angle ABE = AEO = ACB; that is, GBE = ACD, and likewise BGE = ADC, whence CAD = BEG. Fig. 48.

Now let BG be the force acting at B, in direction BA, perp. to CD; then (Prop. IX.) the forces against the sides AC, AD, will be in the directions EB, FB; and therefore EB, EG will represent these forces (by Prop. VIII.), when they keep one another in equilibrio. Therefore force BG applied to the back of the wedge, is to the force BE, perp. to the side AC; as BG to BE; that is, (by similar triangles) as CD to CA.

Cor. 1. *The power acting against the back at* B, *is to that part of the force against* AC, *which acts parallel to the back* CD; *as the back* CD, *is to the hight* AB.

For divide the whole force BE into the two BO, OE; the part EO acts parallel to CD; therefore the force acting at B, is to the force in direction OE or BC; as BG to OE; that is, (by similar triangles) as CD to AB.

Cor. 2. *By reason of the great friction of the wedge, the power at* B, *must be to the resistance against one side* AC; *at least as twice the base* CD, *to the side* AC, *taking the resistance perp. to* AC. *Or as twice the base* CD, *to the hight* AB, *for the resistance parallel to the base* CD; *to overcome the resistance. But the power must be doubled for the resistance against both sides.*

For since the wedge retains any position it is driven into; therefore the friction must be at least equal to the power that drives it.

Cor.

Fig. 48. Cor. 3. *If you reckon the resistance at both sides of the wedge; then, if there is an equilibrium, the power at* B, *is to the whole resistance; as the back* CD, *to the sum of the sides,* CA, AD, *reckoning the resistance perp. to the sides. Or as the back* CD, *to twice the hight* AB, *for the resistance parallel to the back* CD.

This follows directly from the Prop. and Cor. 1.

SCHOLIUM.

The principal use of the wedge is for the cleaving of wood or separating the parts of hard bodies, by the blow of a mallet. The force impressed by a mallet is vastly great in comparison of a dead weight. For if a wedge, which is to cleave a piece of wood, be pressed down with never so great a weight, or even if the other mechanical powers be applied to force it in; yet the effect of them will scarce be sensible; and yet the stroke of a sledge or mallet will force it in. This effect is owing very much to the quantity of motion the mallet is put into, which it communicates in an instant to the wedge, by the force of percussion. A great deal of the resistance is owing to friction, which hinders the motion of the wedge; but the stroke of a mallet overcomes it; upon which account the force of percussion is of excellent use; for a smart stroke puts the body into a tremulous vibrating motion, by which the parts are disunited and separated; and by this means the friction or sticking is overcome, and the motion of the wedge made easy.

This mechanic power is the simplest of any; and to this, may be reduced all edge tools, as knives, axes, chissels, scissars, swords, files, saws, spades, shovels, &c. which are so many wedges fastened to a handle. And also all tools or instruments

ments with a ſharp point, as nails, bodkins, nee- Fig.
dles, pins; and all inſtruments to cleave, cut, ſlit, 48.
chop, pierce, bore, and the like. And in general all inſtruments that have an edge or point.

This Prop. is the ſame as Prop. XXX. in my large book of Mechanics, but demonſtrated after a different way; and both come to the ſame thing, which evinces the truth thereof.

In this Prop. I have ſhewn under what circumſtance, the wedge is in equilibrio; and that is, when the power is to the force againſt either ſide; as the back, is to that ſide. Therefore it muſt be very ſtrange, that any body ſhould underſtand it, as if I had ſaid, that the power is to the whole reſiſtance; as the back, is to one ſide only. They that do this muſt be blind or very careleſs.

Fig.

SECT. V.

The comparative Strength of Beams of Timber, and the Stress they sustain. The Powers of Engines, their Motions, and Friction.

PROP. LIV.

89\. *If a beam of wood* AB, *whose section is a parallelogram, be supported at the ends* A *and* B, *by two props* C, D. *And a weight* E *be laid on the middle of it, to break it; the strength of it will be as the square of the depth* EF, *when the breadth is given.*

For divide the depth EF into an infinite number of equal parts at *n*, *o*, *p*, *q*, *r*, &c. Now the strength of the beam consists of the strength of all the fibres F*n*, *no*, *op*, &c. And to break these fibres, is to break the beam. Also when the beam is stretched by the weight, the fibres *Fn*, *no*, *op*, &c. are stretched by the power of the bended levers AEF, AE*n*, AE*o*, &c. whose support is at E, and power at A. For the pressure at A being half the weight E, we must suppose that pressure applied to A, to overcome the resistances at F, *n*, *o*, &c. Put the force or pressure at A = P, then P acts against all the fibres at F, *n*, *o*, &c. by help of the bended levers AEF, AE*n*, AE*o*, &c. But it is a known property of springs, fibres, and such like expanding bodies; that the further they are stretched, the greater force they exert, in proportion to the length. Therefore when the beam breaks; that is, when the tension of the fibre F*n* is is at its utmost extent; then those in the middle between F and E will have but half the tension, and those at all other distances, will have a tension proportional to that distance. This being settled, let the utmost tension of nF be $= 1$; then the tensions at n, o, p, &c. will be $\frac{En}{EF}$, $\frac{Eo}{EF}$, $\frac{Ep}{EF}$ &c. and the several forces, these exert against the point A, by means of the bended levers FEA, nEA, oEA, &c. will be $\frac{FE}{EA}$, $\frac{En^2}{EF \times EA}$, $\frac{Eo^2}{EF \times EA}$, $\frac{Ep^2}{EF \times EA}$ &c. and the sum of all is $= \frac{1}{EF \times EA} \times$ into $EF^2 + En^2 + Eo^2 + Ep^2$ &c. to o. But (Arith. Inf. Prop. III.) the sum of the progression $EF^2 + En^2 + Eo^2$ &c. to o, is $\frac{1}{3} EF^3$. Therefore the sum of all the forces exerted at A; that is, $P = \frac{1}{EF \times EA} \times \frac{1}{3} EF^3 = \frac{EF^2}{3EA}$. But $P = \frac{1}{2}$ weight E, therefore weight $E = \frac{2}{3} \times \frac{EF^2}{EA}$, when the beam breaks. Fig. 49.

In like manner for any other depth Ep, the weight e that would break it is $= \frac{2Ep^2}{3EA}$. Whence the weight E to the weight e, is as EF^2 to Ep^2; that is, as the squares of the depths; for $\frac{2}{3EA}$ is a given quantity. Therefore the strength of the beams, are as the squares of the depths.

Cor. 1. *Hence the strengths of several pieces of the same timber, are to one another as the breadths and squares of the depths.*

For by this Prop. they are as the squares of the depths when the breadth is given. And if the breadth be increased in any proportion, it is evi-

Fig. dent the ſtrength is increaſed in the ſame propor-
49 tion. So that a beam of the ſame depth being twice as broad is twice as ſtrong, and thrice as broad is thrice as ſtrong, &c.

50. Cor. 2. *If ſeveral beams of timber as AF of the ſame length, ſtick out of a wall; their ſtrength to bear any weight* W *ſuſpended at the end, is as the breadth and ſquare of the depth.*

This follows from Cor. 1. only turning the beam upſide down, to make the weight W ſuſpended at A act downwards inſtead of preſſing upwards.

49. Cor. 3. *If ſeveral pieces of timber be laid under one another, they will be no ſtronger, than if they were laid ſide by ſide.*

For not being connected together in one ſolid piece, they can only exert each its own ſtrength, which will be the ſame in any poſition.

Cor. 4. *Hence the ſame piece of timber is ſtronger when laid edgeways or with the flat ſide up and down, than when laid flat ways, or with the flat ſide horizontal; and that in proportion of the greater breadth to the leſſer.*

For let B be the greater breadth, or the breadth of the flat ſide; *b* the leſſer breadth, being the narrow ſide. Then the ſtrength edge ways is B B*b*, and flat ways B*bb*; and they are to one another as B to *b*.

PROP. LV.

49. *If a beam of timber* AB *be ſupported at both ends; and a given weight* E *laid on the middle of it; the ſtreſs it ſuffers by the weight, will be as its length* AB.

For half the weight E is ſupported at A, by the prop C; and the preſſure at C is equal to it. And this preſſure is always the ſame whatever length AB

AB is of. But it was shewn in the last Prop. that Fig.
the pressure at A, breaks the fibres F*n*, *no*, *op*, &c. 49.
by means of the bended levers AEF, AE*n*, AE*o*, &c. But (by Prop. XLV.) when the lengths EF, E*n*, E*o*, &c. are given, and the power at A also given; the effect at F, *n*, *o*, &c. is so much greater, as the arm AE is longer; that is, the stress at the section EF, is proportional to the distance AE, or to the length of the beam AB.

Cor. 1. *If* AF *be a beam sticking out of a wall, and a weight* W *hung at the end of it. The stress it suffers by the weight, at any point* G, *will be as the distance* AG. 50.

For this has the same effect, as in the case of this Prop. only turning the beam upside down. Or thus, suppose AHG to be a bended lever, whose fulcrum is H; then since GH is given, and the weight W; therefore by the power of the lever, the longer AH is, the more force is applied at G, or any other points, in GH, to separate the parts of the wood; and therefore the stress is as AG.

Cor. 2. *Therefore, instead of a weight, if any force be applied at the end* A, *of the lever* AF; *the stress at any part* G, *will be as the force, and distance* AG.

For augmenting the force, the stress is increased in the same ratio.

Cor. 3. *Hence also if any weight lie upon the middle of a horizontal beam; the stress there will be as the weight and length of the beam.* 49.

For if the weight be increased, the stress will be increased proportionally, all other circumstances remaining the same.

Cor. 4. *The stress of beams by their own weight, will be as the squares of the lengths.*

For here the weight is as the length.

Fig.

PROP. LVI.

51. *If* AB *be a beam of timber whose length is given; and supported at the ends* A *and* B; *and if a given weight* W *be placed at any point of it* G. *The stress of the beam at* G, *will be as the rectangle* AGB.

Let the given weight be W, then (Cor. 3. Prop. XLV.) the weight W is equal to the pressure at both A and B. And (Cor. 2. Prop. XXXI.) pressure at A : pressure at B : : BG : AG, and press. A : pres. A + pres. B : : BG : BG + AG; that is, pres. A : weight W : : BG : AB, therefore pressure at A $= \frac{BG}{AB}$W, and this is the force re-acting at A. But (Prop. LV. Cor. 2.) the stress at G by this force acting at the distance AG, is as the force multiplied by AG; that is, as $\frac{AG \times BG}{AB} \times W$. But W and AB are given, and therefore the stress at G is as AG $\times$ GB.

Cor. 1. *The greatest stress of a beam is when the weight lies in the middle.*

For the greatest rectangle of the parts, is in that point.

Cor. 2. *The stress at any point* P *by a weight at* G; *is equal to the stress at* G, *by the same weight at* P.

For when the weight is at W, the stress at G is AG $\times$ GB, and the stress at P $= \frac{BP}{BG} \times$ the stress at G $= \frac{BP}{BG} \times AG \times GB = BP \times AG$. Again, when the weight is at P, the stress at P is AP $\times$ PB; and the stress at G $= \frac{AG}{AP} \times$ last stress $= \frac{AG}{AP} \times AP \times PB = AG \times PB$, the same as before.

PROP.

Fig.

PROP. LVII.

If the distance of the walls AD *and* BC *be given, and* AB, AC *be two beams of timber of equal thickness; the one horizontal, the other inclined. And if two equal weights* P, Q, *be suspended in the middle of them; the stress is equal in both, and the one will as soon break as the other, by these equal weights.* 52.

For (Prop. XIV.) AC : AB : : weight P : $\frac{AB}{AC}$ P = pressure against the plane, or the part of the weight the beam AC sustains. And (Cor. 3. Prop. LV.) the stress upon AC is $\frac{AB}{AC}$ P × AC or AB × P; and the stress on AB is Q × AB, which is equal to AB × P, because the weights P, Q are equal. Therefore, the stress being the same, and the beams being of equal thickness, one will bear as much as the other, and they will both break together.

Cor. 1. *If the beams be loaded with weights in any other places in the same perpendicular line as* F, G; *they will bear equal stress, and one will as soon break as the other.*

For they are cut into parts similar to one another; and therefore stress at F : stress by P : : AFC : $\frac{1}{4}$ AC² : : AGB : $\frac{AB^2}{4}$: : stress by B : stress by Q or stress by P. Therefore stress at F = stress at B.

Cor. 2. *If the two beams be loaded in proportion to their lengths; the stress by these weights, or by their own weights, will be as their lengths; and therefore the longer, that stands aslope, will sooner break.*

 For

Fig. 52. For the ſtreſs upon AC was AB × P, and the ſtreſs on AB was AB × Q; but ſince P and Q are to one another as AC and AB, therefore the ſtreſs on AC and AB will be as AB × AC and AB × AB; that is, as AC to AB. And in regard to their own weights, theſe are alſo proportional to their lengths.

PROP. LVIII.

53. *Let* AB, AC, *be two beams of timber of equal length and thickneſs, the one horizontal the other ſet ſloping. And if* CD *be perp. to* AB, *and they be loaded in the middle with two weights* P, Q, *which are to one another as* AC *to* AD. *Then the ſtreſs will be equal in both, and one will as ſoon break as the other.*

For (Prop. XIV.) AC : AD : : P : $\frac{AD}{AC}$ P = preſſure of P in the middle of AC. And by ſuppoſition, AC : AD : : P : Q; therefore $\frac{AD}{AC}$ P = Q, the weight in the middle of AB. Therefore the forces in the middle of the two beams are the ſame; and the lengths of the beams being the ſame, therefore (Prop. LV.) the ſtreſs is equal upon both of them; and being of equal thickneſs, if one breaks the other will break.

Cor. *If the weights* P, Q, *be equal upon the two equal beams* AB, AC. *The ſtreſs upon* AB *will be to the ſtreſs upon* AC, *as* AB *or* AC *to* AD. *The ſame holds in regard to their own weights.*

For the weight Q is increaſed in that proportion.

SCHOLIUM.

Many more propoſitions relating to the ſtrength of timber might be inſerted; as for example, if a weight

weight was disposed equally thro' the length of Fig.
the beam AB (fig. 51.), supported at both ends; 53.
the stress in any point G, is as the rectangle AGB. And the stress at any point G is but half of the stress it would suffer, if the whole weight was suspended at G. Also if AF (fig. 50.) be a beam fixed in a wall at one end, and a weight be dispersed uniformly thro' all the length of it. The stress at any point G, with that weight (or with its own weight, if it be all of a thickness), will be as AG square, the square of the distance from the end. And the stress at any point G by a weight suspended at A, will be double the stress at the same point G, when the same weight is dispersed uniformly thro' the part AG. They that would see these and such like things demonstrated, may consult my large book of Mechanics, to which I refer the reader.

PROP. LIX.

If several pieces of timber be applied to any mechanical use where strength is required; not only the parts of the same piece, but the several pieces in regard to one another, ought to be so adjusted for bigness; that the strength may be always proportional to the stress they are to endure.

This Prop. is the foundation of all good Mechanism, and ought to be regarded in all sorts of tools and instruments we work with, as well as in the several parts of any engine. For who that is wise, will overload himself with his work tools, or make them bigger and heavier than the work requires? neither ought they to be so slender as not to be able to perform their office. In all engines, it must be considered what weight every beam is to carry, and proportion the strength accordingly. All levers

Fig. vers muſt be made ſtrongeſt at the place where they are ſtrained the moſt; in levers of the firſt kind, they muſt be ſtrongeſt at the ſupport. In thoſe of the ſecond kind, at the weight. In thoſe of the third kind, at the power, and diminiſh proportionally from that point. The axles of wheels and pullies, the teeth of wheels, which bear greater weights, or act with greater force, muſt be made ſtronger. And thoſe lighter, that have light work to do. Ropes muſt be ſo much ſtronger or weaker, as they have more or leſs tenſion. And in general, all the parts of a machine muſt have ſuch a degree of ſtrength as to be able to perform its office, and no more. For an exceſs of ſtrength in any part does no good, but adds unneceſſary weight to the machine, which clogs and retards its motion, and makes it languid and dead. And on the other hand, a defect of ſtrength where it is wanted, will be a means to make the engine fail in that part, and go to ruin. So neceſſary it is to adjuſt the ſtrength to the ſtreſs, that a good mechanic will never neglect it; but will contrive all the parts in due proportion, by which means they will laſt all alike, and the whole machine will be diſpoſed to fail all at once. And this will ever diſtinguiſh a good mechanic from a bad one, who either makes ſome parts ſo defective, imperfect and feeble as to fail very ſoon; or makes others, ſo ſtrong or clumſey, as to out laſt all the reſt.

From this general rule follows

Cor. 1. *In ſeveral pieces of timber of the ſame ſort, or in different parts of the ſame piece; the breadth multiplied by the ſquare of the depth, muſt be as the length multiplied by the weight to be born.*

For then the ſtrength will be as the ſtreſs.

Cor. 2. *The breadth mutiplied by the ſquare of the depth, and divided by the product of the length and weight, muſt be the ſame in all.* Cor.

Cor. 3. *Hence may be computed the strength of timber proper for several uses in building. As,* Fig.

1. To find the dimensions of joists and boards for flooring. Let b, d, l be the breadth, depth and length of a joist, n = number of them, x = their distance, g = depth of a board, w = weight; then $nbdd$ = strength of all the joists, and wl = stress on them, also $nlgg$ = strength of the boards, and wx their stress; therefore $\frac{nbdd}{wl} = \frac{nlgg}{wx}$; and $x = \frac{llgg}{bdd}$, for the distance of the joists, or the length of a board between them. Or $b = \frac{llgg}{ddx}$, or $dd = \frac{llgg}{bx}$, and so on, according to what is wanted.

2. To find the dimensions of square timber for the roof of a house. Let r, s, l be the length of the ribs, spars and lats, so far as they bear; x, y, z their breadth or depth, n the distances of the lats, w = weight upon a rib, c = cosine of elevation of the roof. Then by reason of the inclined plane, $\frac{lw}{r} \times c$ = weight upon a spar. And $\frac{lnw}{rs}$ = weight upon a lat: for the ribs and lats lie horizontally. Therefore (Cor. 2.) $\frac{x^3}{wr} = \frac{y^3}{\frac{slw}{r} \times c} = \frac{z^3}{l \times \frac{lnw}{rs}}$.

Whence $x^3 = \frac{rry^3}{cls}$, and $x^3 = \frac{rrsz^3}{lln}$. Hence if any one x, y, or z be given, and all the rest of the quantities; the other two may be found. Or in general, any two being unknown, they may be found, from having the rest given.

For example, let r = 9 feet, s = 4 feet, l = 15 inches, n = 11 inches, c = .707 the cosine of 45°, the pitch of the roof. And assume $y = 2\frac{1}{2}$ inches;

Fig. inches; then $x = 2\frac{1}{2}\sqrt[3]{\frac{81}{3.535}} = 7.1$ inches. And

$z = y\sqrt[3]{\frac{ln}{css}} = 2\frac{1}{2}\sqrt[3]{\frac{55}{543}} = 1\frac{1}{6}$ inches.

54. 3. To find the curve ACB, into the form of which, if a joiſt be cut, on the upper or under ſide; and having the two ſides parallel planes, which are perp. to the horizon. That the ſaid joiſt ſhall be equally ſtrong every where to bear a given weight, ſuſpended on it.

Let the weight be placed in the ordinate CD; and the breadth of the beam, and the weight being given; then (Prop. LIV.) the ſtrength at C is as CD^2. And (Prop. LVI.) the ſtreſs is as ADB. Therefore that the ſtrength may be as the ſtreſs, CD^2 is as the rectangle ADB; and therefore the curve ACB is an ellipſis.

55. 4. To find the figure of a beam AB, fixed with one end in a wall, and having a given weight W ſuſpended at the other end B; and being every where of the ſame depth; it may be equally ſtrong throughout.

Let CD be the breadth at C; then (Prop. LIV.) the ſtrength is as CD. And (Cor. 1. Prop. LV.) the ſtreſs is as CB. Therefore CD is every where as CB, and therefore CDB is a plane triangle. And the beam is a priſm, whoſe upper and under ſides are parallel to the horizon.

56. 5. To find the figure of a beam AB, ſticking with one end in a wall, and of a given breadth; having a weight W ſuſpended at the end B; ſo that it may be equally ſtrong throughout.

Let CD be the depth at C. Then ſince the breadth is given, the ſtrength is as CD^2. And the ſtreſs as DB; therefore CD^2 is as DB. Whence CD is a common parabola.

6. To find the figure of a beam AB, of the
57. ſame breadth and depth, ſticking in a wall with one

one end, and bearing a weight suspended at the other end B; so that it may be equally strong throughout. Fig. 57.

Let CD be the thickness at O. Then the strength is as CD^3, and the stress is as BO. Therefore BO is as CD^3 or as CO^3. And consequently ACB is a cubic parabola, whose vertex is at B.

7. In like manner, if CBD be a beam fixed with one end in a wall, and all the sides of it be cut into the form of a concave parabola, whose vertex is at B. It will be equally strong throughout for supporting its own weight. 58.

For putting $BO = x$, $CO = y$, then by nature of the curve, $ay = xx$. But the solidity of CBD is $\frac{3.1416yyx}{5}$. And the center of gravity I, is distant $\frac{5}{6}x$ from B, therefore $OI = \frac{1}{6}x$. Now CD^3 or $8y^3 =$ strength at O. And CBD × OI or $\frac{3.1416yyx}{5} \times \frac{1}{6}x =$ stress. Therefore the strength : to the stress : : is as $8y^3$: to $\frac{3.1416y^2xx}{30}$: : $8y$: $\frac{3.1416xx}{30}$: : $8y$: $\frac{3.1416ay}{30}$: : 340 : $3.1416a$, that is, in a given ratio. And as this happens every where, the solid is equally strong in all parts.

I must take notice here that the 116th figure in my large book of Mechanics, is drawn wrong. It should be concave instead of being convex.

8. Again, if AB be the spire of a church which is a solid cone or pyramid; it will be equally strong throughout for resisting the wind. For the quantity of wind falling on any part of it ACD, will be as the section ACD. Therefore let $AO = x$, $CD = y$. And $x = ay$, then the strength at $O = y^3$, and if I be the center of gravity of ACD, then OI 59.

Fig. 59. $OI = \frac{1}{3}x$. And the ſtreſs at O = wind $ACD \times OI = xy \times \frac{1}{3}x$. Therefore the ſtrength is to the ſtreſs : : as $y^3 : \frac{1}{3}xxy : : yy : \frac{1}{3}xx$ or $\frac{1}{3}aayy : : 3 : aa$; that is, in a given ratio. Therefore the ſpire is equally ſtrong every where.

SCHOLIUM.

It is all along ſuppoſed that the timber, &c. is of equal goodneſs, where theſe proportions for ſtrength are made. But if it is otherwiſe, a proper allowance muſt be made for the defect.

In theſe Propoſitions, I have called every thing *Strength*, that contributes in a direct proportion to reſiſt any force acting againſt a beam to break it; and I call *Streſs*, whatever weakens it in a direct proportion. But the whole may be referred to the article of ſtrength; for what I have called ſtreſs may be reckoned ſtrength in an inverſe ratio. Thus the ſtrength of a piece of timber may be ſaid to be directly as the breadth and ſquare of the depth, and inverſely as its length, and the weight or force applied; and that is equivalent to taking in the ſtreſs. But I had rather keep them diſtinct, and refer to each of them their proper effects, as I have along done in the foregoing examples.

A piece of wood a foot long, and an inch ſquare, will bear as follows; oak from 320 to 1100; elm from 310 to 930; fir from 280 to 770 pounds, according to the goodneſs.

PROP.

PROP. LX.

In any machine contrived to raiſe great weights; if the power applied, be to the weight to be raiſed; as the velocity of the weight, to the velocity of the power; the power will only be in equilibrio with the weight. Therefore to raiſe it, the power muſt be ſo far increaſed, as to overcome all the friction and reſiſtance ariſing from the engine or otherwiſe; and then the power will be able to raiſe the weight.

A man would be much miſtaken, who ſhall make an engine to raiſe a great weight, and give his power no greater velocity, in regard to the velocity of the weight; than the quantity of the weight has in regard to the quantity of the power. For when he has done that, his weight and power will but have equal quantities of motion, and therefore they cannot ſet one another a moving, but muſt always remain at reſt. It is neceſſary then, that he do one of theſe two things. 1. That he apply a power greater than in that proportion, ſo much as to overcome all the friction and other accidental reſiſtance that may happen: and in ſome engines theſe are very great. Or 2. He muſt ſo continue his engine, that the velocity of the power, which ſuppoſe he has given, may be ſo much greater than the velocity of the weight; as the quantity of the weight, friction, and reſiſtance and all together, is greater than the power. This being done, the greater power will always overcome the leſſer, and his engine will work.

If a man does not attend to this rule, he will be guilty of many abſurd miſtakes, either in attempting things that are impoſſible, or in not applying means proper for the purpoſe Hence it is that engines contrived for mines and water-works ſo often

Fig. ten fail; as they muſt when either the quantity or velocity of the power is too little; or which is the ſame thing, when the velocity of the weight is too great, and therefore would require more power than what is propoſed. As the weight is to move ſlow, the conſequence is, that it will be ſo much a longer time in moving thro' any ſpace. But there is no help for that. For as much as the weight to be raiſed is the greater, the time of raiſing it will be ſo much greater too.

Cor. 1. *Hence in raiſing any weight, what is gained in power is loſt in time. Or the time of riſing thro' any hight will be ſo much longer as the weight is greater.*

If the power be to the weight as 1 to 20, then the ſpace thro' which the weight moves will be 20 times leſs, and the time will be 20 times longer in moving thro' any ſpace, than that of the power. The advantage that is gained by the ſtrength of the motion, is loſt in the ſlowneſs of it. So that tho' they increaſe the power, they prolong the time. And that which one man may do in 20 days, may be done by the ſtrength of twenty men in one day.

Cor. 2. *The quantity of motion in the weight is not at all increaſed by the engine. And if any given quantity of power be immediately applied to a body at liberty, it will produce as much motion in it, as it would do by help of a machine.*

PROP. LXI.

If an engine be compoſed of ſeveral of the ſimple mechanic powers combined together; it will produce the ſame effect, ſetting aſide friction; as any one ſimple mechanic power would do, which has the ſame power or force of acting.

For let any compound engine be divided into all the ſimple powers that compoſe it. Then the force

or

or power applied to the firſt part, will cauſe it to act upon the ſecond with a new power, which would be deemed the weight, if the machine had no more parts. This new power acting on the ſecond part, will cauſe it to act upon the third part; and that upon a fourth, and ſo on till you come at the weight, which will be acted on, by all theſe mediums, juſt the ſame as by a ſimple machine whoſe power is equal to them all. Fig.

Cor. 1. *Hence a compound machine may be made, which ſhall have the ſame power, as any ſingle one propoſed.*

For if a lever is propoſed whoſe power is 100 to 1; two levers acting on one another will be equivalent to it, where the power of the firſt is as 10 to 1, and that of the ſecond alſo as 10 to 1, or the firſt 20 to 1, and the ſecond 5 to 1; or any two numbers, whoſe product is 100.

Again, a wheel and axle whoſe power is as 48 to 1, may be reſolved into two or more wheels with teeth, to have the ſame power; for example, make two wheels, ſo that the firſt wheel and pinion be as 8 to 1, and the ſecond as 6 to 1. They will have the ſame effect as the ſingle one. Or break it into three wheels, whoſe ſeveral powers may be 4 to 1, and 4 to 1, and 3 to 1.

If a ſimple combination of pullies be as 36 to 1; you may take three combinations to act upon one another, whoſe powers are 3 to 1, 3 to 1, and 4 to 1.

And after the ſame manner it is to be done in machines more compounded.

And this is generally done to ſave room. For when an engine is to have great power, it is hardly made of one wheel, it would be ſo large; but by breaking it into ſeveral wheels, after this manner; it will go into a little room, and have the

Fig. ſame power as the other. All the inconvenience is, it will have more friction; for the more parts acting upon one another, the more friction is made.

Cor. 2. *Hence alſo it follows, that in any compound machine, its power is to the weight, in the compound ratio of the power to the weight in all the ſimple machines that compoſe it.*

Cor. 3. *Hence it will be no difficult matter to contrive an engine that ſhall overcome any force or reſiſtance aſſigned.*

For if you have the quantity of power given, as well as of the weight or reſiſtance; it is but taking any ſimple machine as a lever, wheel, &c. ſo that the power may be to the weight in the ratio aſſigned, adding as much to the weight as you judge the friction will amount to. When this ſimple machine is obtained; break it or reſolve it into as many other ſimple ones as you think proper; ſo that they may have the ſame power.

And as to the ſeveral ſimple machines, it matters not what ſort they are of, as to the power; whether they be levers, wheels, pullies, or ſcrews; but ſome are more commodious than others for particular purpoſes; which a mechanic will find out beſt by practice. In general, a lever is the moſt ready and ſimple machine to raiſe a weight a ſmall diſtance; and for further diſtances, the wheel and axle, or a combination of pullies; or the perpetual ſcrew. Alſo theſe may be combined with one another; as a lever with a wheel or a ſcrew, the wheel and axle with pullies, pullies with pullies, and wheels with wheels, the perpetual ſcrew and the wheel. But in general a machine ſhould conſiſt of as few parts as is conſiſtent with the purpoſe it is deſigned for, upon account of leſſening the friction; and to make it ſtill leſs, the joints muſt be

be oiled or greaſed. All parts that act on one Fig.
another muſt be poliſhed ſmooth. The axles or ſpindles of wheels muſt not ſhake in the holes, but run true and even. Likewiſe the larger a machine is, if it be well executed, the better and truer it will work. And large wheels and pullies, and ſmall axles or ſpindles have the leaſt friction.

The power applied to work the engine may be men or horſes; or it may be weight or a ſpring; or wind, water, or fire; of which one muſt take that which is moſt convenient and coſts the leaſt. Wind and water are beſt applied to work large engines, and ſuch as muſt be continually kept going. A man may act for a while againſt a reſiſtance of 50 pounds; and for a whole day againſt 30 pounds. A horſe is about as ſtrong as five men.

If two men work at a roller, the handles ought to be at right angles to one another.

When a machine is to go regular and uniform, a heavy wheel or fly muſt be applied to it.

SCHOLIUM.

Two things are required to make a good engineer. 1. A good invention for the ſimple and eaſy contrivance of a machine, and this is to be attained by practice and experience. 2. So much theory as to be able to compute the effect any engine will have; and this is to be learned from the principles of Mechanics.

PROP. LXII.

The friction or reſiſtance ariſing by a body moving upon any ſurface, is as the roughneſs of the ſurface, and nearly as the weight of the body; but is not much increaſed by the quantity of the ſurface of the moving body, and is ſomething greater with a greater velocity.

It is matter of experience that bodies meet with a great deal of reſiſtance by ſliding upon one ano-

Fig. ther, which cannot be entirely taken away, tho' the bodies be made never ſo ſmooth: yet by ſmoothing or poliſhing their ſurfaces, and taking off the roughneſs of them, this reſiſtance may be reduced to a ſmall matter. But many bodies, by their natural texture, are not capable of bearing a poliſh; and theſe will always have a conſiderable degree of reſiſtance or friction. And thoſe that can be poliſhed, will have ſome of this reſiſtance ariſing from the coheſion of their ſurfaces. But in general, the ſmoother or finer their ſurfaces, the leſs the friction will be.

As the ſurfaces of all bodies are in ſome degree rough and uneven, and ſubject to many inequalities; when one body is laid upon another, the prominent parts of one fall into the hollows of the other; ſo that the body cannot be moved forward, till the prominent parts of one be raiſed above the prominent parts of the other, which requires the more force to effect, as theſe parts are higher; that is, as the body is rougher. And this is ſimilar to drawing a body up an inclined plane, for theſe protuberances are nothing elſe but ſo many inclined planes, over which the body is to be drawn. And therefore the heavier the body, the more force is required to draw it over theſe eminencies; whence the friction will be nearly as the weight of the body.

But whilſt the roughneſs remains the ſame, or the prominent parts remain of the ſame hight, there will always be required the ſame force, to draw the ſame weight. And the increaſing of the ſurface, retaining the ſame weight, can add nothing to the reſiſtance on that account; but it will make ſome addition upon other accounts. For when one ſurface is dragged along another, ſome part of the reſiſtance ariſes from ſome parts of the moving ſurface, taking hold of the parts of the other, and tearing them off; and this is called *wearing*. And there-

Fig.
therefore this part of the friction is greater in a greater surface, in proportion to that surface. There is likewise in a greater surface, a greater force of cohesion, which still adds something to the friction. But the two parts of the friction, arising from the wearing and tenacity, are not increased by the velocity: but the other part, of drawing them over inclined planes, will increase with the velocity. So that in the whole, the friction is something increased by the quantity of the surface, and by the velocity, but not much. But more in some bodies than others, according to their particular texture.

Cor. 1. *Hence there can be no certain rule, to estimate the friction of bodies; this is a matter that can only be decided by experiments. But it may be observed, that, ceteris paribus, hard bodies will have less resistance than softer; and bodies oiled or greased, will have far less.*

For the particles of hard bodies, cannot so well take hold of one another to tear themselves off. And when a surface is oiled, it is the same thing as if it run upon a great number of rollers or spheres.

Cor. 2. *Hence also a method appears of measuring the friction of a body sliding upon another body, by help of an inclined plane.*

Take a plank CB of the same matter, raise it at 69. one end C so high, till the body whose friction is sought, being laid at C, shall just begin to move down the plane CB. Then the weight of the body, is to the friction as the base AB, to the hight AC of the plane. For the pressure against the plane is the part of the weight that causes the friction, and the tendency down the plane is equal to the friction. And (Prop. XIV.) that pressure is to the tendency as AB to AC.

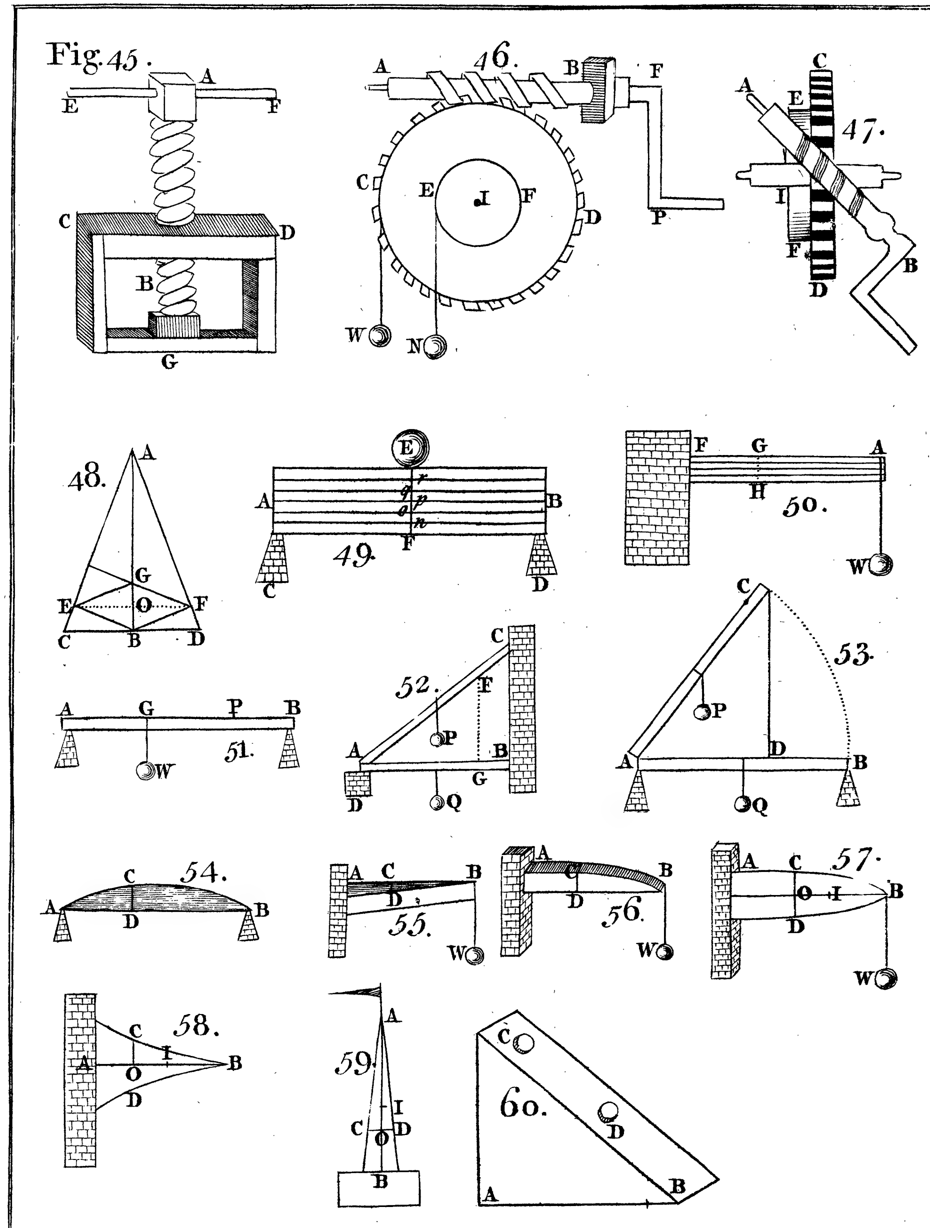
Fig. 60. If you puſh the body from C downward, and obſerve it to keep the ſame velocity thro' D to B; then you will have the friction for that velocity. If it increaſes its velocity, lower the end of the plank C; if it grows ſlower, raiſe the end C, till you get the body to have the ſame velocity quite thro' the plane. And ſo you will find what elevations are proper for each velocity; and from thence the ratio of AB to AC, or of the weight to the friction.

There is a way to make the experiment, by drawing the body along a horizontal plane, by weights hung at a ſtring, which goes over a pulley; but the method here deſcribed is more eaſy and ſimple.

Scholium.

From what has been before laid down, it will be eaſy to underſtand the nature of engines, and how to contrive one for any purpoſe aſſigned. And likewiſe having any engine before us, we can by the ſame rules, compute its powers and operations.

Engines are of various kinds; ſome are fixed in a particular place, where they are to act; as wind-mills and water-mills for corn, fire engines for drawing water, gins for coal pits, many ſorts of mills; pumps, cranes, &c. others are movable from one place to another, and may be carried to any place where they are wanted, as blocks, pullies and tackles for raiſing weights, the lifting jack, and lifting ſtock, clocks, watches, ſmall bellows, ſcales, ſteelyards, and an infinite number of others. Another ſort of engines are ſuch as are made on purpoſe to move from one place to another, ſuch as boats, ſhips, coaches, carriages, waggons, &c. If any of theſe are urged forward by the help of levers, wheels, &c. By having the acting power given, the moving force that drives it forward, is eaſily found by the properties of theſe machines. Only obſerve, if the firſt acting power



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power be external, as wind, water, horses, &c. Fig.
you must not forget to add or subtract it, to or from the moving force before found; according as that first acting power conspires with, or opposes the motion of the machine; and the result is the true force it is driven forward with. I have only room to describe a very few engines, but those that desire it may see great variety in my large book of Mechanics.

A WHEEL CARRIAGE.

AB is a cart or carriage, going upon two wheels 61.
as CD, and sometimes upon four, as all waggons do. The advantages of wheel carriages is so great, that no body who has any great weight to carry, will make use of any other method. Was a great weight to be dragged along upon a sledge or any such machine without wheels, the friction would be so great, that a sufficient force in many cases could not be got to do it. But by applying wheels to carriages, the friction is almost all of it taken away. And this is occasioned by the wheels turning round upon the ground, instead of dragging upon it. And the reason of the wheel's turning round is the resistance the earth makes against it at O where it touches. For as the carriages goes along, the wheel meets with a resistance at the bottom O, where it touches the ground; and meeting with none at the top at C, to balance it; that force at O must make it turn round in the order ODC, so that all the parts of the circumference of the wheel are successively applied to the earth. In going down a steep bank it is often necessary to tie one wheel fast, that it cannot turn round, this will make it drag; and by the great resistance it meets with, stops the too violent motion, the carriage would otherwise have, in descending the hill.

Fig. 61. But altho' all ſorts of wheels very much diminiſh the friction; yet ſome have more than others; and it may be obſerved that great wheels, and ſmall axles have the leaſt friction. To make the friction as little as poſſible, ſome have applied friction wheels, which is thus; EG is the friction wheel running upon an axis I which is fixed in the piece of timber ES, which timber is fixed to the ſide of the carriage. KL is the axle of the carriage, which is fixed in the wheel CD, ſo that both turn round together. Then inſtead of the carriage lying upon the axle KL, the friction wheel FG lies upon the axle; ſo that when the wheel CD turns round, the axle cauſes the friction wheel, with the weight of the carriage upon it, to turn round the center I, which diminiſhes the friction in proportion to the radius IG: and there is the ſame contrivance for the wheel on the other ſide. But the wheel CD need not be fixed to the axle; for it may turn round on the axle KL, and alſo the axle turn round under the carriage.

In paſſing over any obſtacles, the large wheels have the advantage. For let MN be an obſtacle; then drawing the wheel over this obſtacle, is the ſame thing as drawing it up the inclined plane MP, which is a tangent to the point M; but the greater the wheel CD is, the leſs is that plane inclined to the horizon.

Likewiſe great wheels do not ſink ſo deep into the earth as ſmall ones, and conſequently require leſs force to pull them out again.

But there are diſadvantges in great wheels; for in the firſt place, they are more eaſily overturned; and ſecondly, they are not ſo eaſy to turn with, in a ſtrait road as ſmall wheels.

The tackle of any carriage ought to be ſo fixed, that the horſe may pull partly upwards, or lift, as well as pull forwards; for all hills and inequalities

in

in the road, being like so many inclined planes, Fig.
the weight is most easily drawn over them, when 61.
the power draws at an equal elevation.

A carriage with four wheels is more advantageous, than one with two only, but they are bad to turn; and therefore are obliged to make use of small fore-wheels. Broad wheels which are lately come into fashion, are very advantageous, as they sink but little into the earth. But there is a disadvantage attends them, for they take up such a quantity of dirt by their great breadth, as sensibly retards the carriage by its weight, and the like may be said of their own weight.

The under side of the axle where the wheels are, must be in a right line; otherwise if they slant upwards, the weight of the carriage will cause them to work toward the end, and press against the runners and lin pin. And as the ends of the axle are conical, this causes the wheels to come nearer together at bottom, and be further distant at the top; by which means the carriage is sooner overturned. To help this, the ends of the axle must be made as near a cylindrical form as possible, to get the wheels to fit, and to move free.

A HAND MILL.

Fig. 62. is a hand mill for grinding corn, A, B 62.
the stones included in a wooden case. A the upper stone, being the living or moving stone. B the lower stone, or the dead stone, being fixed immovable. The upper stone is 5 inches thick, and a foot and three quarters broad; the lower stone is broader. C is a cog-wheel, with 16 or 18 cogs; DE its axis. F is a trundle with 9 rounds, fixed to the axis G, which axis is fixed to the upper stone A, by a piece of iron made on purpose. H is the hopper, into which the corn is put; I the shoe, to carry the corn by little and little thro' a hole at

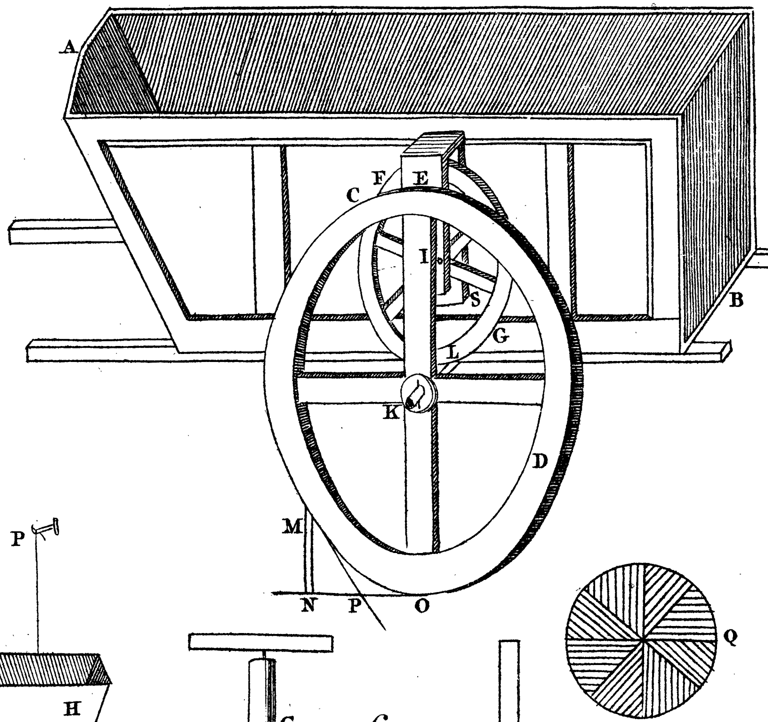
K,

Fig. 62. K, to fall between the two ſtones. L is the mill eye, being the place where the flour or meal comes out after it is ground. The under ſtone is ſupported by ſtrong beams not drawn here. And the ſpindle G ſtands on the beam MN, which lies upon the bearer O, and O lies upon a fixed beam at one end, and at the other end has a ſtring fixed, and tied to the pin P. The under ſtone is not flat, but riſes a little in the middle, and the upper one is a little hollow. The ſtones very near touch at the out ſide, but are wider towards the middle to let the corn go in.

When corn is to be ground, it is put into the hopper H, a little at a time, and a man turns the handle D, which carries round the cog-wheel C, and this carries about the trundle F, and axis G, and ſtone A. The axis G is angular at K; and as it goes round, it ſhakes the ſhoe I, and makes the corn fall gradually thro' the hole K. And the upper ſtone going round grinds it, and when ground it comes out at the mill eye L, where there is a ſack or tub placed to receive it. Another handle may be made at E like that at D, for two men to work, if any one pleaſes. In order to make the mill grind courſer or finer, the upper ſtone A may be lowered or raiſed, by means of the ſtring going from the bearer O; for turning round the pin P, the ſtring is lengthened or ſhortened, and thereby the timbers O, M are lowered or raiſed, and with them the axle G and ſtone A. For the ſpindle G goes thro' the ſtone B, and runs upon the beam MN. The ſpindle is made ſo cloſe and tight, by wood or leather, where it goes thro the under ſtone, that no meal can fall thro'. The under ſide of the upper ſtone is cut into gutters in the manner repreſented at Q. It is a pity ſome ſuch like mills are not made at a cheap rate for the ſake of the poor, who are much diſtreſſed by the roguery of the millers.

Fig.

Fig. 61.



62.

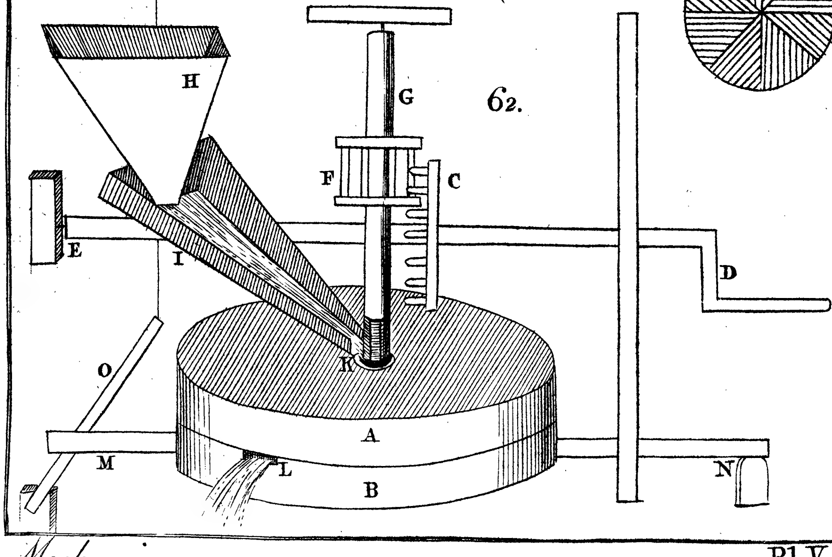


Fig. 63. is a sort of crane, BC an upright post, AB a beam fixed horizontally at top of it; these turn round together on the pivot C, and within the circle S, which is fixed to the top of the frame PQ. EF is a wooden roller, or rather a roller made of thin boards, for lightness, and all nailed to several circular pieces on the inside. GH a wheel fixed to the roller, about which goes the rope GR. IK, LN, two other ropes; fixed with one end to the cross piece AB, and the other end to the roller EF. W a weight equal to the weight of the wheel and roller, which is fastened to a rope which goes over the pulley O, and then is fastened to a collar V, which goes round the roller. ET is another rope with a hook at it to lift up any weight, the other end of the rope being fixed to the roller; here are in all five ropes. Fig. 63.

To raise any weight as M, hang it upon the hook T, then pulling at the rope R which goes about the wheel GH, this causes the wheel and roller to turn round, and the ropes IK, LN to wind about it, by which means the wheel and axle rises; and by rising, folds the rope TE about the roller the contrary way, and so raises the weight M. When the weight M is raised high enough, a man must take hold of the rope T with a hook, by which the whole machine may be drawn about, turning upon the centers C and S. And then the weight M may be let down again. The weight of the wheel and roller do not affect the power drawing at R, because it is balanced by the weight W. There is no friction in this machine but what is occasioned by the collar V, and the bending of the ropes. And the power is to the weight in this crane, as the diameter of the roller to the radius of the wheel GH.

Fig.

An ENGINE for raising WEIGHTS.

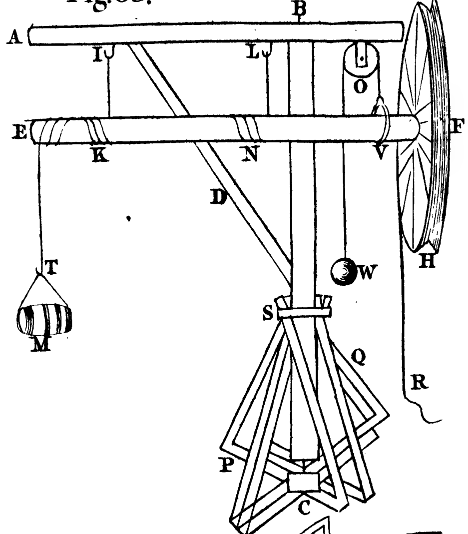
64. Fig. 64. is an engine compoſed of a perpetual ſcrew AB, and a wheel DE with teeth, and a ſingle pulley H. FG is an axle, about which a rope goes, which lifts the pulley and weight W. BC is the winch, to turn it round withal. As the ſpindle AB is turned about, the teeth of it takes the teeth of the wheel DE, and turns it about, together with the axle FG, which winds up the rope, and raiſes the pulley H, with the weight W. The power at C, is to the weight W, as diameter FG × by the breadth of one tooth, is to twice the diameter DE × circumference of the circle deſcribed by C.

A FULLING MILL.

65. Fig. 65. is a fulling mill. AB a great water wheel, carried about by a ſtream of water, coming from the trough C, and falling into the buckets D, D, D whoſe weight carries the wheel about; this is a breaſt mill, becauſe the water comes no higher than the middle or breaſt of the wheel; EF is its axis; I, I; K, K, two lifters going thro' the axle, which raiſe the ends G, G of the wooden mallets GH, GH, as the wheel goes about; and when the end G ſlips off the cog or lifter K or I, the mallet falls into the trough L, and each of the mallets makes two ſtrokes for one revolution of the wheel. The mallets move about the centers M, M. Theſe troughs L, L, contain the ſtuff which is to be milled, by the beating of the mallets. N, N, is a channel to carry the water, being juſt wide enough to let the wheel go round. And the wheel may be ſtopt, by turning the trough C aſide, which brings the water. In this engine more mallets may be uſed, and then more pins or lifters muſt be put thro' the axis EF.

Fig.

Fig. 63.


A
B
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W
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M
Q
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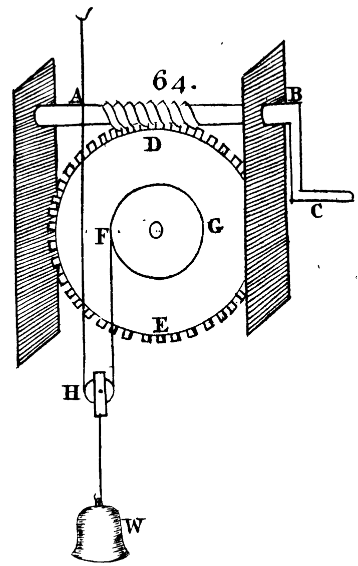
C

64.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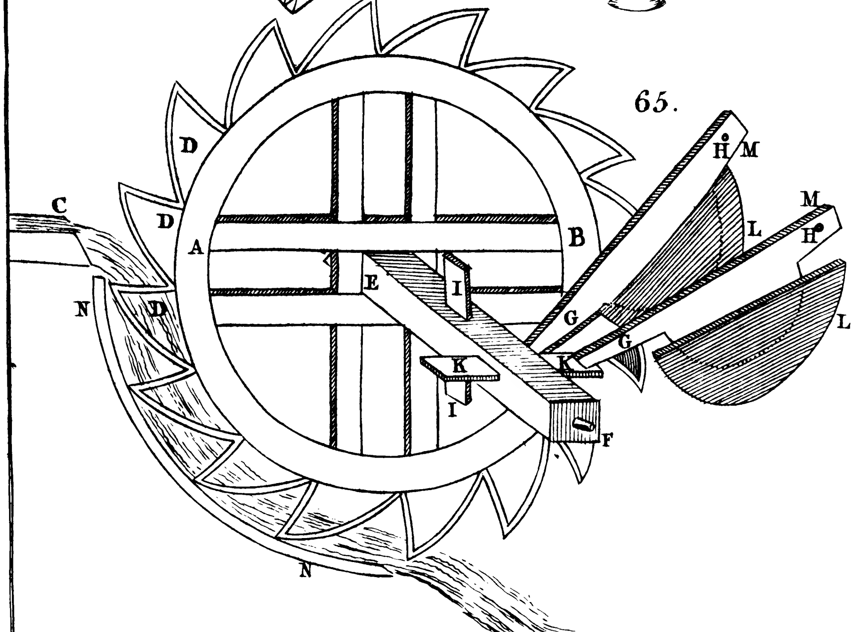
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Fig. 66. is a common *Pocket Watch.* AA the *balance*, BB the *verge*; C, C, two *palats*. D the *crown wheel* acting againſt the palats C, C; E its pinion. F the *contrate wheel*, G its pinion. H the *third wheel*, I its pinion. K the ſecond wheel or *center wheel*, L its pinion. M the *great wheel*, N the *fuſee* turning round upon the ſpindle of M. O the *ſpring box*, having a ſpring included in it. PP the *chain* going round the ſpring box O, and the fuſee N. This work is within the watch between the two plates. Here the face is downward, and in the watch the wheel K is placed in the center, and the others round about it. Here I have placed them ſo as beſt to be ſeen, which ſignifies nothing to the motion. The balance AA is without the plate, covered by the cock X. The minute hand Q goes upon the axis of the wheel K. Fig. 66.

Then between the upper plate and the face, we have V the *cannon pinion* or *pinion of report*. Z the *dial wheel*. T the *minute wheel*. S the *pinion* or *nut*, fixed to it. The ſocket of the cannon pinion V goes into the ſocket of the wheel Z, and are movable about one another, and both go thro' the face; on the ſocket of the pinion Z is fixed the hour hand R; and on the ſocket of V is fixed the minute hand Q. Likewiſe the ſocket of V is hollow, and both go upon the arbour of the wheel K, which reaches thro' the face, and are faſtened there. The wheel and ſocket, T, S are hollow, and go upon a fixed axle on which they turn round.

When the chain PP is wound up, upon the fuſee N; the ſpring included in the box O, draws the chain PP, which forces about the wheel M, the fuſee being kept from ſlipping back, by a catch on purpoſe. Then M drives L, and K, and K drives I, and H drives G, and F drives E, and the teeth of the crown wheel D, act againſt the palats C, C alternately, and cauſe the balance AA to vibrate

Fig. vibrate back and forward, and thus the watch is kept going.

66. The cannon pinion, and dial wheel V and Z, and the hands Q, R, being put upon the arbor of K at W; and faſtened there, by means of a ſhoulder which is upon the axis, and a braſs ſpring; as the wheel K goes round, it carries with it the pinion V with the minute hand, and V drives T together with S; and S drives Z with the hour hand.

The numbers of the wheels and pinions, (that is the teeth in them) are, M = 48, L = 12, K = 54, I = 6, H = 48, G = 6, F = 48, E = 6, D = 15, and 2 palats. The *train*, or number of beats in an hour, is 17280, which is about $4\frac{3}{4}$ beats in a ſecond. Alſo V = 10, Z = 36, S = 12, T = 40.

The wheel M goes round 6 times in 24 hours, therefore K goes round $\left(\frac{48}{12}\right)$ 4 times as much; that is, 24 times, or once in an hour, and the hand Q along with it; therefore Q will ſhew minutes. Then as V goes round once in an hour, T will go round $\left(\frac{10}{40}\right)$ $\frac{1}{4}$ of that, or $\frac{1}{4}$ the circumference; and as S goes $\frac{1}{4}$, Z will go $\left(\frac{12}{36}\right)$ $\frac{1}{3}$ of that, or $\frac{1}{12}$ of the circumference in an hour, and therefore as R goes along with it, R will ſhew the hours. The wheels and pinions T, Z, and S, V, are drawn with the face upwards. And the whole machine included in a caſe is but about two inches diameter. There is a ſpiral ſpring fixed under the balance AB, called the *regulator*, which gives it a regular motion; and likewiſe abundance of ſmall parts helpful to her motion, too long to be deſcribed here.

The

The way of writing down the numbers, is thus, Fig. 66.

48
12 — 54
6 — 48
6 — 48
6 — 15
2

10 Q
40 — 12
36 R.

Explanation. The wheel with 48 drives a pinion of 12, and a wheel of 54 on the ſame arbor. The wheel 54 drives the pinion 6 with the wheel 48 on the ſame arbor. The wheel 48 drives the pinion 6 and wheel 48 on the ſame arbor. The wheel 48 drives the pinion 6 and wheel 15 on the ſame arbor. And the wheel 15 drives the two palats.

Again the wheel 54 has the pinion 10 on its arbor, and the hand Q; and the pinion 10 drives the wheel 40, with the pinion 12. And the pinion 12 drives the wheel 36 with the hand R.

As this machine is moved by a ſpring, it is ſubject to very great inequalities of motion, occaſioned by heat and cold. For hot weather ſo relaxes, ſoftens, and weakens the main ſpring, that it loſes a great deal of its ſtrength, which cauſes the watch to loſe time and go too ſlow. On the other hand, cold froſty weather ſo affects the ſpring, and it is ſo condenſed and hardened, that it becomes far ſtronger; and by that means accelerates the motion of the watch, and makes her go faſter. The difference of motion in a watch, thus occaſioned by heat and cold, will often amount to an hour, and more in 24 hours. To remedy this, there is a piece of machinery, called the *Slide*, placed near the regulating ſpring; which being put forward or backward, ſhortens or lengthens the ſpring, ſo as to make her keep time truly.

Some people have been ſo ſilly as to think, that the greater ſtrength of a ſpring ariſes wholly from

Fig. from its being made ſhorter, as this happens to be one of the effects of cold. But it is eaſily demon-
67. ſtrated that this is not the cauſe. For let AB be a ſpring as it is dilated by heat, and *ab* the ſame ſpring contracted by cold. Now if the ſpring has been contracted in length, it muſt be proportionally contracted in all dimenſions. Let *l*, *b*, *d*, denote the length, breadth, and depth, in its cold, and leaſt dimenſions; and *rl*, *rb*, *rd*, the length, breadth, and depth, in its hot and greateſt dimenſions. Then (Prop. LIV.) the ſtrength of the longer, to the ſtrength of the ſhorter, will be as $\frac{rb \times rrdd}{rl}$ to $\frac{bdd}{l}$ (conſidering it weakened by the length), and that is as rr to 1, or as AB^2 to ab^2. So that the longer ſpring, upon account of its being affected with heat, is ſo far from being weaker, than the ſhorter affected with cold, that it is the ſtronger of the two. And therefore this difference is not to be aſcribed merely to the lengthning or ſhortning thereof; but muſt be owing to the nature, texture and conſtitution of the ſteel, as it is ſome way or other affected and changed by the heat and cold.

And that there is ſome change induced by the cold, into the very texture of the mettal, is evident from this, that all ſorts of tools made of iron or ſteel, as ſprings, knives, ſaws, nails, &c. very eaſily ſnap and break in cold froſty weather, which they will not do in hot weather. And that property of ſteel ſprings is the true cauſe, that theſe ſorts of movements can never go true.

66. To make a calculation of the different forces requiſite to make a watch gain or loſe any number of minutes, as ſuppoſe half an hour in 24; and I have often experienced it to be more. By Cor. 4. Prop. VI. the product of the force and ſquare of the

the time, is as the product of the body and ſpace Fig. deſcribed, which here is a given quantity. For 66. the matter of the balance remains the ſame in hot as cold weather; and ſo does the length of the ſwing, which here is the ſpace deſcribed. Therefore the force is reciprocally as the ſquare of the time of vibrating, or directly as the ſquare of the number of vibrations in 24 hours. Therefore the force with the warm ſpring, is to the force with the cold one; as the ſquare of 23 $\frac{1}{2}$ hours, to the ſquare of 24; that is, nearly as 23 to 24. So that if a ſpring was to contract half an inch in a foot in length, without altering its other dimenſions, it would but be ſufficient to account for that phænomenon; but this is forty times more than the lengthening and ſhortening by heat and cold, for that does not alter ſo much as a thouſandth part, as is plain from experiments.

The caſe being thus, a clock or watch going by a ſpring, can never be made to keep time truly, except it be always kept to the ſame degree of heat or cold, which cannot be done without conſtant attendance. And if any ſort of mechaniſm be contrived to correct this; yet as ſuch a thing can only be made by gueſs, it cannot be truſted to at ſea, but only for ſhort voyages. But no motion however regular, can ever anſwer at ſea, where the irregular motion of the ſhip will continually diſturb it; add to this, that the ſmall compaſs a watch is contained in, makes it eaſier diſturbed, than a larger machine would be; but to ſuppoſe that any regular motion can ſubſiſt among ten thouſand irregular motions, and in ten thouſand different directions, is a moſt glaring abſurdity And if any one with ſuch a machine would but make trial of it to the Eaſt Indies, he would find the abſurdity and diſappointment. And therefore I never expect to ſee ſuch a time keeper, or any ſuch thing as a watch or clock

Fig. 66. going by a ſpring, to keep true time at ſea. But time will diſcover all things.

As to pendulum clocks, their irregularity in the ſame latitude is owing to nothing but the lengthning or ſhortning of the pendulum; which is a mere trifle to the other. But then they would be infinitely more diſturbed at ſea, than a watch; and in a ſtorm could not go at all. In different latitudes too, another irregularity attends a pendulum, depending on the different forces of gravity. Tho' this amounts but to a ſmall matter, yet it makes a conſiderable variation, in a great length of time. For in ſouth latitudes, where the gravity is leſs, a clock loſes time. And in north latitudes, where the gravity is greater, it gains time. So that none of theſe machines are fit to meaſure time at ſea, altho' ten times ten thouſand pounds ſhould be given away for making them.

A DESCENDING CLOCK.

68. Fig. 68. is a clock deſcending down an inclined plane. This conſiſts of a train of watch work, contained between two circular plates AB, CD, 4 inches diameter, fixed together by a hoop an inch and half broad, incloſing all the work. The inner work conſiſts of 5 wheels, the ſame as in a watch, only there is a ſpur wheel inſtead of the contrate wheel, as 4; *b* is the balance, whoſe palats play in the teeth of the crown wheel 5. Here is no ſpring to give it motion, but inſtead thereof, the weight W is fixed to the wheel 1, and ſo adjuſted for weight, that it may balance the lower ſide, and hinder it from rolling down the plane. Now whilſt the weight W moves the wheel 1, this wheel by moving about, cauſes the weight W to deſcend, by which it ceaſes to be a balance for the oppoſite ſide, and therefore that ſide begins to deſcend,

ſcend, till the weight W be raiſed high enough again to become a balance, which muſt be about the poſition it appears in the figure. Thus whilſt wheels move gradually about, the weight W deſcends gradually, which makes the body of the machine turn gradually round, and deſcend down the inclined plane PQ; making one revolution in 12 hours. And therefore to have her to go 24 or 30 hours; the length of the plane PQ muſt be 2 or $2\frac{1}{2}$ circumferences of the plates. Before the weight W is fixed to the wheel 1, ſome lead or braſs muſt be ſoldered on the ſide E oppoſite to the wheels 2, 3, 4, &c. for the wheel 1 muſt be in the center. And then the lead or braſs muſt be filed away till the center of gravity of the machine be in the center of the plates. And to hinder the machine from ſliding, the edges of the plates muſt be lightly indented. The inclined plane PQ may be a board, which muſt be elevated 10 or 12 degrees, but that is to be found by trials; for if ſhe go too ſlow the end P muſt be raiſed; but if too faſt it muſt be lowered. When the clock has gone the length of the board to Q, it muſt be ſet again at P. The fore ſide CD is divided into hours, and a pin is fixed in the center at G, on which the hand FGH, always hangs looſely in a perp. poſition, with the heavy end H downward. And the end F ſhews the hour of the day. So that the hours come to the hand, and not the hand to the hours. Fig. 68.

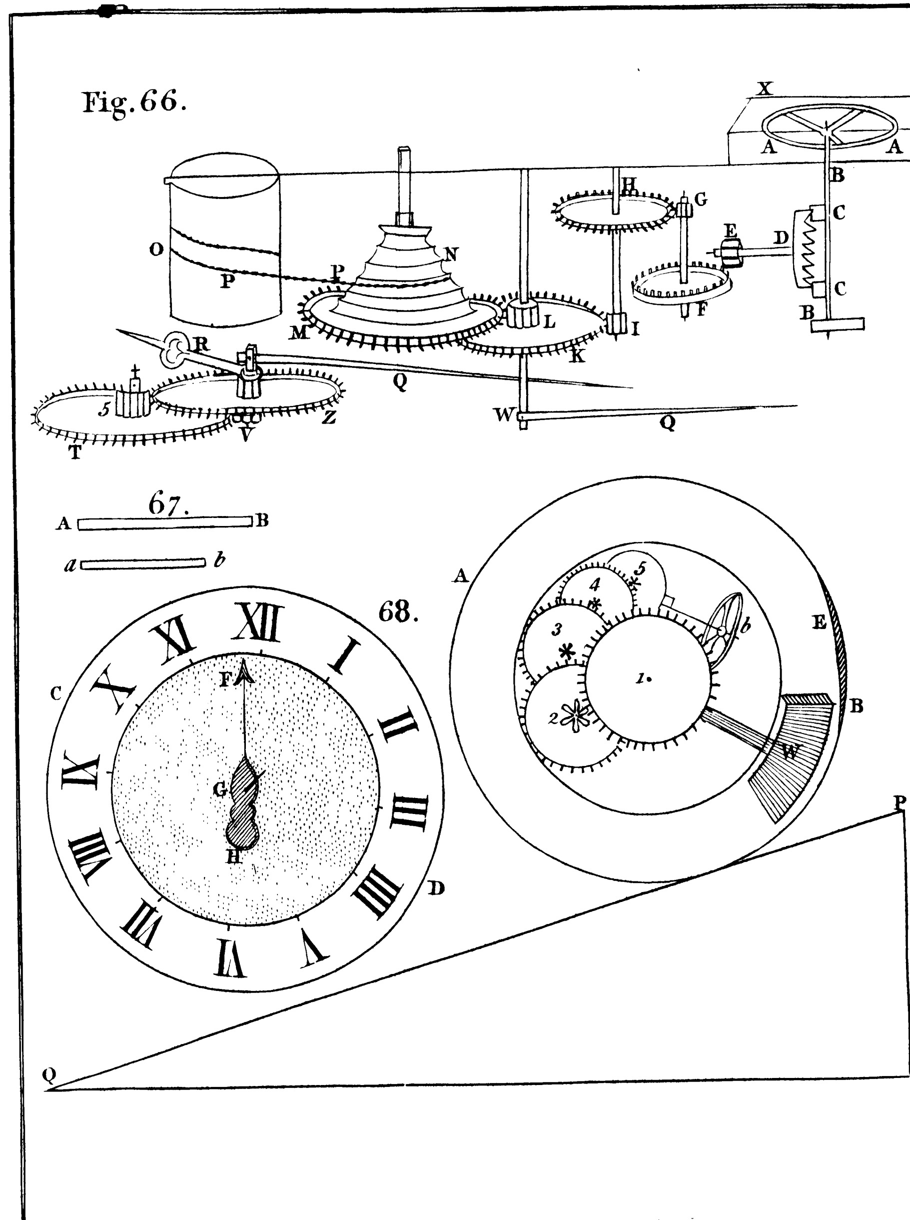
The board PQ muſt be be perfectly ſtreight from one end to the other, or elſe ſhe will go faſter in ſome places, and ſlower in others.

The circle with hours ought to be a narrow rim of braſs, movable round about, by the help of of one or more pins placed in it; ſo that it may be ſet to the true time.

 The

Fig. 68. The weight W ſerves for two uſes, 1, to be a counterpoiſe to the ſide A; and 2, by its weight to put the clock in motion.

The weight W muſt be ſo heavy as to make the clock keep time, when it has a proper degree of elevation as 45 degrees; and then the board muſt have an elevation of 10 or 12 degrees. If ſhe go too faſt, with theſe poſitions, take ſome thing off the weight; if too ſlow, add ſomething to it.



Mechanics Pl. VII. pa. 116.

Fig.

SECT. VI.

HYDROSTATICS *and* PNEUMATICS.

DEFINITION I.

A Fluid, is ſuch a body whoſe parts are eaſily moved among themſelves, and yield to any force acting againſt them; but resume their places upon its removal; else one might imagine flour & Dust to be Fluid.

DEF. II.

Hydroſtatics, is a ſcience that demonſtrates the properties of fluids.

DEF.. III.

Hydraulics, is the art of raiſing water by engines.

DEF. IV.

Pneumatics, is that ſcience which ſhews the properties of the air.

DEF. V.

A fountain or *jet d'eau*, is an artificial ſpout of water.

PROP. LXIII.

If one part of a fluid be higher than another, the higher parts will continually deſcend to the lower places, and will not be at reſt, till the ſurface of it is quite level.

For the parts of a fluid being movable every way, if any part is above the reſt, it will deſcend by its own gravity as low as it can get. And afterwards other parts that are now become higher, will

Fig. will defcend as the other did, till at laft they will all be reduced to a level or horizontal plane.

Cor. 1. *Hence water that communicates by means of a channel or pipe, with other water; will fettle at the fame hight in both places.*

Cor. 2. *For the fame reafon, if a fluid gravitates towards a center; it will difpofe itfelf into a fpherical figure, whofe center is the center of force. As the fea in refpect of the earth.*

PROP. LXIV.

If a fluid be at reft in a veffel whofe bafe is parallel to the horizon; equal parts of the bafe are equally preffed by the fluid.

For upon every part of the bafe there is an equal column of the fluid fupported by it. And as all thefe columns are of equal weight, they muft prefs the bafe equally; or equal parts of the bafe will fuftain an equal preffure.

Cor. 1. *All parts of the fluid prefs equally at the fame depth.*

For imagine a plain drawn thro' the fluid parallel to the horizon. Then the preffure will be the fame in any part of that plane, and therefore the parts of the fluid at the fame depth fuftain the fame preffure.

Cor. 2. *The preffure of a fluid at any depth, is as the depth of the fluid.*

For the preffure is as the weight, and the weight is as the hight of a column of the fluid.

PROP.

Fig.

PROP. LXV.

If a fluid is compressed by its weight or otherwise; at any point it presses equally, in all manner of directions.

This arises from the nature of fluidity; which is, to yield to any force in any direction. If it cannot give way to any force applied, it will press against other parts of the fluid in direction of that force. And the pressure in all directions with be the same. For if any one was less, the fluid would move that way, till the pressure be equal every way.

Cor. *In any vessel containing a fluid; the pressure is the same against the bottom, as against the sides, or even upwards, at the same depth.*

PROP. LXVI.

The pressure of a fluid upon the base of the containing vessel, is as the base, and perpendicular altitude; whatever be the figure of the vessel that contains it. 69.

Let ABIC, EGKH be two vessels. Then (Prop. LXIV. Cor. 2.) the pressure upon an inch on the base AB = hight CD × 1 inch. And the pressure upon an inch on the base HK is = hight FH × 1 inch. But (Prop. LXIV.) equal parts of the bases are equally pressed, therefore the pressure on the base AB is CD × number of inches in AB; and pressure on the base HK is FH × number of inches in HK. That is, the pressure on AB is to the pressure on HK; as base AB × hight CD, to the base HK × hight FH.

Cor. 1. *Hence if the hights be equal, the pressures are as the bases. And if both the hights and bases be equal;*

Fig. 69. *equal; the pressures are equal in both; tho' their contents be never so different.*

For the reason that the wider vessel EK, has no greater pressure at the bottom, is, because the oblique sides EH, GK, take off part of the weight. And in the narrower vessel CB, the sides CA, IB, re-act against the pressure of the water, which is all alike at the same depth; and by this re-action the pressure is increased at the bottom, so as to become the same every where.

Cor. 2. *The pressure against the base of any vessel, is the same as of a cylinder of an equal base and hight.*

70. Cor. 3. *If there be a recurve tube* ABF, *in which are two different fluids* CD, EF. *Their hights in the two legs* CD, EF, *will be reciprocally as their specific gravities, when they are at rest.*

For if the fluid EF be twice or thrice as light as CD; it must have twice or thrice the hight, to have an equal pressure, to counterbalance the other.

PROP. LXVII.

71. *If a body of the same specific gravity of a fluid; be immersed in it, it will rest in any place of it. A body of greater density will sink; and one of a less density will swim.*

Let A, B, C be three bodies; whereof A is lighter bulk for bulk than the fluid; B is equal; and C heavier. The body B, being of the same density, or equal in weight as so much of the fluid; it will press the fluid under it just as much as if the space was filled with the fluid. The pressure then will be the same all around it, as if the fluid was there, and consequently there is no force to put it out of its place. But if the body be lighter, the

the preſſure of it downwards will be leſs than before; and leſs than in other places at the ſame depth; and conſequently the leſſer force will give way, and it will riſe to the top. And if the body be heavier, the preſſure downwards will be greater than before; and the greater preſſure will prevail and carry it to the bottom. Fig. 71.

Cor. 1. *Hence if ſeveral bodies of different ſpecific gravity be immerſed in a fluid; the heavieſt will get the loweſt.*

For the heavieſt are impelled with a greater force, and therefore will go faſteſt down.

Cor. 2. *A body immerſed in a fluid, loſes as much weight, as an equal quantity of the fluid weighs. And the fluid gains it.*

For if the body is of the ſame ſpecific gravity as the fluid; then it will loſe all its weight. And if it be lighter or heavier, there remains only the difference of the weights of the body and fluid, to move the body.

Cor. 3. *All bodies of equal magnitudes, loſe equal weights in the ſame fluid. And bodies of different magnitudes loſe weights proportional to the magnitudes.*

Cor. 4. *The weights loſt in different fluids, by immerging the ſame body therein, are as the ſpecific gravities of the fluids. And bodies of equal weight, loſe weights in the ſame fluid, reciprocally as the ſpecific gravities of the bodies.*

Cor. 5. *The weight of a body ſwimming in a fluid, is equal to the weight of as much of the fluid, as the immerſed part of the body takes up.*

For the preſſure underneath the ſwimming body is juſt the ſame as ſo much of the immerſed fluid; and therefore the weights are the ſame.

Cor.

Fig. 71. Cor. 6. *Hence a body will ſink deeper in a lighter fluid than in a heavier.*

Cor. 7. *Hence appears the reaſon why we do not feel the whole weight of an immerſed body, till it be drawn quite out of the water.*

PROP. LXVIII.

72. *If a fluid runs thro' a pipe, ſo as to leave no vacuities; the velocity of the fluid in different parts of it, will be reciprocally as the tranſverſe ſections, in theſe parts.*

Let AC, LB be the ſections at A and L. And let the part of the fluid ACBL come to the place *acbl*. Then will the ſolid ACBL = ſolid *acbl*; take away the part *ac*BL common to both; and we have AC*ca* = LB*bl*. But in equal ſolids the baſes and hights are reciprocally proportional. But if D*f* be the axis of the pipe, the hights D*d*, F*f*, paſſed thro' in equal times, are as the velocities. Therefore, ſection AC : ſection LB : : velocity along F*f* : velocity along D*d*.

PROP. LXIX.

73. *If* AD *is a veſſel of water or any other fluid*; B *a hole in the bottom or ſide. Then if the veſſel be always kept full; in the time a heavy body falls thro' half the hight of the water above the hole* AB, *a cylinder of water will flow out of the hole, whoſe hight is* AB, *and baſe the area of the hole.*

The preſſure of the water againſt the hole B, by which the motion is generated, is equal to the weight of a column of water whoſe hight is AB, and baſe the area B (by Cor. 2. Prop. LXVI.). But equal forces generate equal motions; and ſince a cylinder

cylinder of water falling thro' $\frac{1}{2}$ AB by its gravity, acquires ſuch a motion, as to paſs thro' the whole hight AB in that time. Therefore in that time the water running out muſt acquire the ſame motion. And that the effluent water may have the ſame motion, a cylinder muſt run out whoſe length is AB; and then the ſpace deſcribed by the water in that time will alſo be AB, for that ſpace is the length of the cylinder run out. Therefore this is the quantity run out in that time. Fig. 73.

Cor. 1. *The quantity run out in any time is equal to a cylinder or priſm, whoſe length is the ſpace deſcribed in that time by the velocity acquired by falling thro' half the hight, and whoſe baſe is the hole.*

For the length of the cylinder is as the time of running out.

Cor. 2. *The velocity a little without the hole, is greater than in the hole; and is nearly equal to the velocity of a body falling thro' the whole hight* AB.

For without the hole the ſtream is contracted by the water's converging from all ſides to the center of the hole. And this makes the velocity greater in about the ratio of 1 to $\sqrt{2}$.

Cor. 3. *The water ſpouts out with the ſame velocity, whether it be downwards, or ſideways, or upwards. And therefore if it be upwards, it aſcends nearly to the hight of the water above the hole.*

Cor. 4. *The velocities and likewiſe the quantities of the ſpouting water, at different depths; will be as the ſquare roots of the depths.*

SCHOLIUM.

From hence are derived the rules for the conſtruction of fountains or jets. Let ABC be a reſervoir of water, CDE a pipe coming from it, to bring 74.

Fig. bring water to the fountain which ſpouts up at E,
74. to the hight EF, near to the level of the reſervoir AB. In order to have a fountain in perfection, the pipe CD muſt be wide, and covered with a thin plate at E with a hole in it, not above the fifth or ſixth part of the diameter of the pipe CD. And this pipe muſt be curve having no angles. If the reſervoir be 50 feet high, the diameter of the hole at E may be an inch, and the diameter of the pipe 6 inches. In general, the diameter of the hole E, ought to be as the ſquare root of the hight of the reſervoir. When the water runs thro' a great length of pipe, the jet will not riſe ſo high. A jet never riſes to the full hight of the reſervoir; in a 5 feet jet it wants an inch, and it falls ſhort by lengths which are as the ſquares of the hights; and ſmaller jets loſe more. No jet will riſe 300 feet high.

75. A ſmall fountain is eaſily made by taking a ſtrong bottle A, and filling it half full of water; cement a tube BI very cloſe in it, going near the bottom of the bottle. Then blow in at the top B, to compreſs the air within; and the water will ſpout out at B. If a fountain be placed in the ſunſhine and made to play, it will ſhew all the colours of the rainbow, if a black cloth be placed beyond it.

A jet goes higher if it is not exactly perpendicular; for then the upper part of the jet falls to one ſide without reſiſting the column below. The reſiſtance of the air will alſo deſtroy a deal of its motion, and hinder it from riſing to the hight of the reſervoir. Alſo the friction of the tube or pipe of conduct has a great ſhare in retarding the motion.

78. If there be an upright veſſel as AF full of water, and ſeveral holes be made in the ſide as B, C, D: then the diſtances, the water will ſpout, upon the horizontal plane EL, will be as the ſquare roots of the rectangles of the ſegments, ABE, ACE, and ADE. For the ſpaces will be as the velocities

velocities and times. But (Cor. 4.) the velocity of Fig. the water flowing out of B, will be as $\sqrt{AB}$, and 78. the time of its moving (which is the same as the time of its fall) will be (by Prop. XIII.) as $\sqrt{BE}$; therefore the distance EH is as $\sqrt{AB \times BE}$; and the space EL as $\sqrt{ACE}$. And hence if two holes are made equidistant from top and bottom, they will project the water to the same distance, for if AB = DE, then ABE = ADE, which makes EH the same for both, and hence also it follows, that the projection from the middle point C will be furthest; for ACE is the greatest rectangle. These are the proportions of the distances; but for the absolute distances, it will be thus. The velocity thro' any hole B, will carry it thro' 2AB in the time of falling thro' AB; then to find how far it will move in the time of falling thro' BE. Since these times are as the square roots of the hights, it will be, $\sqrt{AB} : 2AB :: \sqrt{BE} : EH = 2AB\sqrt{\frac{BE}{AB}} = 2\sqrt{ABE}$; and so the space EL = $2\sqrt{ACE}$. It is plain, these curves are parabolas. For the horizontal motion being uniform; EH will be as the time; that is, as $\sqrt{BE}$, or BE will be as EH^2, which is the property of a parabola.

If there be a broad vessel ABDC full of water, 76. and the top AB fits exactly into it; and if the small pipe FE of a great length be soldered close into the top, and if water be poured into the top of the pipe F, till it be full; it will raise a great weight laid upon the top, with the little quantity of water contained in the pipe; which weight will be nearly equal to a column of the fluid, whose base is the top AB; and hight, that of the pipe EF. For the pressure of the water against the top AB, is equal to the weight of that column of water,

Fig. ter, by Prop. LXV. and Cor. And Prop. LXVI.
76. Cor. 2.

But here the tube muſt not be too ſmall. For in capillary tubes the attraction of the glaſs will take off its gravity. If a very ſmall tube be immerſed with one end in a veſſel of water, the water will riſe in the tube above the ſurface of the water; and the higher, the ſmaller the tube is. But in quickſilver, it deſcends in the tube below the external ſurface, from the repulſion of the glaſs.

77. To explain the operation of a ſyphon, which is a crooked pipe CDE, to draw liquors off. Set the ſyphon with the ends C, E, upwards, and fill it with water at the end E till it run out at C; to prevent it, clap the finger at C, and fill the other end to the top, and ſtop that with the finger. Then keeping both ends ſtopt, invert the ſhorter end C into a veſſel of water AB, and take off the fingers, and the water will run out at E, till it be as low as C in the veſſel; provided the end E be always lower than C. Since E is always below C, the hight of the column of water DE is greater than that of CD, and therefore DE muſt out weigh CD and deſcend, and CD will follow after, being forced up by the preſſure of the air, which acts upon the ſurface of the water in the veſſel AB.

The ſurface of the earth falls below the horizontal level only an inch in 620 yards; and in other diſtances the deſcents are as the ſquares of the diſtances.

79. And to find the nature of the curve DCG, forming the jet IDG. Let AK be the hight or top of the reſervoir HF, and ſuppoſe the ſtream to aſcend without any friction, or reſiſtance. By the laws of falling bodies the velocity in any place B, will be as $\sqrt{AB}$. Put the ſemidiameter of the hole at D $= d$, and AD $= h$. Then ſince the ſame water paſſes thro' the ſections at D and B; therefore (Prop. LXVIII.) the velocity will be reciprocally as

as the section; whence $\sqrt{h} : \frac{1}{dd} :: \sqrt{AB} : \frac{1}{BC^2}$; Fig. 79.
therefore $\frac{\sqrt{h}}{BC^2} = \frac{\sqrt{AB}}{dd}$, and $dd\sqrt{h} = BC^2\sqrt{AB}$, whence $AB \times BC^4 = hd^4$; which is a paraboliform figure whose assymptote is AK, for the nature of the cataractic curve DCG. And if the fluid was to descend thro' a hole, as IC; it would form itself into the same figure GCD in descending.

PROP. LXX.

The resistance any body meets with in moving thro' a fluid is as the square of the velocity.

For if any body moves with twice the velocity of another body equal to it, it will strike against twice as much of the fluid, and with twice the velocity; and therefore has four times the resistance; for that will be as the matter and velocity. And if it moves with thrice the velocity, it strikes against thrice as much of the fluid in the same time, with thrice the velocity, and therefore has nine times the resistance. And so on for all other velocities.

Cor. *If a stream of water whose diameter is given, strike against an obstacle at rest; the force against it will be as the square of the velocity of the stream.*

For the reason is the same; since with twice or thrice the velocity; twice or thrice as much of the fluid impinges upon it, in the same time.

PROP. LXXI.

The force of a stream of water against any plane obstacle at rest, is equal to the weight of a column of water, whose base is the section of the stream; and hight, the space descended thro' by a falling body, to acquire that velocity.

For let there be a reservoir whose hight is that space fallen thro'. Then the water (by Cor. 2. Prop.

Fig. Prop. LXIX.) flowing out at the bottom of the reservatory, has the ſame motion as the ſtream; but this is generated by the weight of that column of water, which is the force producing it. And that ſame motion is deſtroyed by the obſtacle, therefore the force againſt it is the very ſame: for there is required as much force to deſtroy as to generate any motion.

Cor. *The force of a ſtream of water flowing out at a hole in the bottom of a reſervatory, is equal to the weight of a column of the fluid of the ſame hight and whoſe baſe is the hole.*

PROP. LXXII. *Prob.*

To find the ſpecific gravity of ſolids or fluids.

1. *For a ſolid heavier than water.*

Weigh the body ſeparately, firſt out of water, and then ſuſpended in water. And divide the weight out of water by the difference of the weights, gives the ſpecific gravity; reckoning the ſpecific gravity of water 1.

For the difference of the weights is equal to the weight of as much water (by Cor. 2. Prop. LXVII.); and the weights of equal magnitudes, are as the ſpecific gravities; therefore the difference of theſe weights; is to the weight of the body; as the ſpecific gravity of water 1, to the ſpecific gravity of the body.

2. *For a body lighter than water.*

Take a piece of any heavy body, ſo big as being tied to the light body, it may ſink it in water. Weigh the heavy body in and out of water, and find the loſs of weight. Alſo weigh the compound both in and out of water, and find alſo the loſs of weight.

weight. Then divide the weight of the light body (out of water), by the difference of these losses, gives the specific gravity; the specific gravity of water being 1. Fig.

For the last loss is = weight of water equal in magnitude to the compound.

And the first loss is = weight of water equal in magnitude to the heavy body.

Whence the dif. losses is = weight of water equal in magnitude to the light body.

and the weights of equal magnitudes, being as the specific gravities; therefore the difference of the losses, (or the weight of water equal to the light body) : weight of the light body : : specific gravity of water 1 : specific gravity of the light body.

3. *For a fluid of any sort.*

Take a piece of a body whose specific gravity you know; weigh it both in and out of the fluid; take the difference of the weights, and multiply it by the specific gravity of the solid body, and divide the product by the weight of the body (out of water), for the specific gravity of the fluid.

For the difference of the weights in and out of water, is the weight of so much of the fluid as equals the magnitude of the body. And the weight of equal magnitudes being as the specific gravities; therefore, weight of the solid : difference of the weights (or the weight of so much of the fluid) : : specific gravity of the solid : to the specific gravity of the fluid.

Example.

I weighed a piece of lead ore, which was 124 grains; and in water it weighed 104 grains, the

Fig. difference is 20; then $\frac{124}{20} = 6.2$; the ſpecific gravity of the ore.

A table of ſpecific gravities.

Fine gold — — —	19.640
Standard gold — — —	18.888
Quickſilver — — —	14.000
Lead — — —	11.340
Fine ſilver — — —	11.092
Standard ſilver — — —	10.536
Copper — — —	9.000
Copper half-pence — —	8.915
Gun metal — — —	8.784
Fine braſs — — —	8.350
Caſt braſs — — —	8.100
Steel — — —	7.850
Iron — — —	7.644
Pewter — — —	7.471
Tin — — —	7.320
Caſt iron — — —	7.000
Lead ore — — —	6.200
Copper ore — — —	5.167
Lapis calaminaris — —	5.000
Load ſtone — — —	4.930
Antimony — — —	4.000
Diamond — — —	3.517
Iſland chriſtal — — —	2.720
Stone, hard — — —	2.700
Rock chriſtal — — —	2.650
Glaſs — — —	2.600
Flint — — —	2.570
Common ſtone — — —	2.500
Chriſtal — — —	2.210
Brick — — —	2.000
Earth — — —	1.984
Horn — — —	1.840

Ivory

Fig.

Ivory — — —	1.820
Chalk — — —	1.793
Allum — — —	1.714
Clay — — —	1.712
Oil of vitriol — — —	1.700
Honey — — —	1.450
Lignum vitæ — — —	1.327
Treacle — — —	1.290
Pitch — — —	1.150
Rozin — — —	1.100
Mohogany — — —	1.063
Amber — — —	1.040
Urine — — —	1.032
Milk — — —	1.031
Brazil — — —	1.031
Box — — —	1.030
Sea water — — —	1.030
Ale — — —	1.028
Vinegar — — —	1.026
Tar — — —	1.015
Common clear water — — —	1.000
Bee wax — — —	.955
Butter — — —	.940
Linseed oil — — —	.932
Brandy — — —	.927
Sallad oil — — —	.913
Logwood — — —	.913
Ice — — —	.908
Oak — — —	.830
Ash — — —	.830
Elm — — —	.820
Oil of turpentine — — —	.810
Walnut tree — — —	.650
Fir — — —	.580
Cork — — —	.238
New fallen snow — — —	.086
Air — — —	.0012

Fig. Cor. 1. *As the weight lost in a fluid, is to the absolute weight of the body; so is the specific gravity of the fluid, to the specific gravity of the body.*

Cor. 2. *Having the specific gravity of a body, and the weight of it; the solidity may be found thus; multiply the weight in pounds by 62 $\frac{1}{2}$. They say as that product to 1; so is the weight of the body in pounds, to the content in feet. And having the content given, one may find the weight, by working backwards.*

For a cubic foot of water weighs 62 $\frac{1}{2}$ lb. averdupoise; and therefore a cubic foot of the body weighs 62 $\frac{1}{2}$ × by the specific gravity of the body. Whence the weight of the body, divided by that product, gives the number of feet in it. Or as 1, to that product; so is the content, to the weight.

SCHOLIUM.

80. The specific gravities of bodies may be found with a pair of scales; suspending the body in water, by a horse hair. But there is an instrument for this purpose called the *Hydrostatical Balance*, the construction of which is thus. AB is the stand and pedestal, having at the top two cheeks of steel, on which the beam CD is suspended, which is like the beam of a pair of scales, and must play freely, and be it self exactly in equilibrio. To this belongs the glass bubble G, and the glass bucket H, and four other parts E, F, I, L. To these are loops fastened to hang them by. And the weights of all these are so adjusted, that E = F + the bubble in water, or = I + the bucket out of water, or = I + L + the bucket in water. Whence L = difference of the weights of the bucket in and out of water. And if you please you may have a weight K, so that K + bubble in water = bubble out of water; or else find it in grains.

grains. The piece L has a slit in it to slip it upon the shank of I. Fig. 80.

It is plain the weight K $=$ weight of water as big as the bubble, or a water bubble.

Then to find the specific gravity of a solid.

Hang E at one end of the balance, and I and the bucket with the solid in it, at the other end; and find what weight is a balance to it.

Then slip L upon I, and immerge the bucket and solid in the water, and find again what weight balances it. Then the first weight divided by the difference of the weights, is the specific gravity of the body; that of water being 1.

For fluids.

Hang E at one end, and F with the bubble at the other; plunge the bubble into the fluid in the vessel MN. Then find the weight P which makes a balance. Then the specific gravity of the fluid is $= \frac{K+P}{K}$, when P is laid on F; or $= \frac{K-P}{K}$, when P is laid on E.

For E being equal to I + the bucket; the first weight found for a balance, is the weight of the solid. Again, E being equal to I + L + the bucket in water; the weight to balance that, is the weight of the solid in water; and the difference, is $=$ to the weight of as much water. Therefore (Cor. 1.) the first weight divided by that difference, is the specific gravity of the body.

Again, since E is $=$ to F + the bubble in water; therefore P is the difference of the weights, of the fluid and so much water; that is, P $=$ difference of K and a fluid bubble; or P $=$ fluid $-$ K, when the fluid is heavier than water, or when P is laid on F. And therefore P $=$ K $-$ the fluid

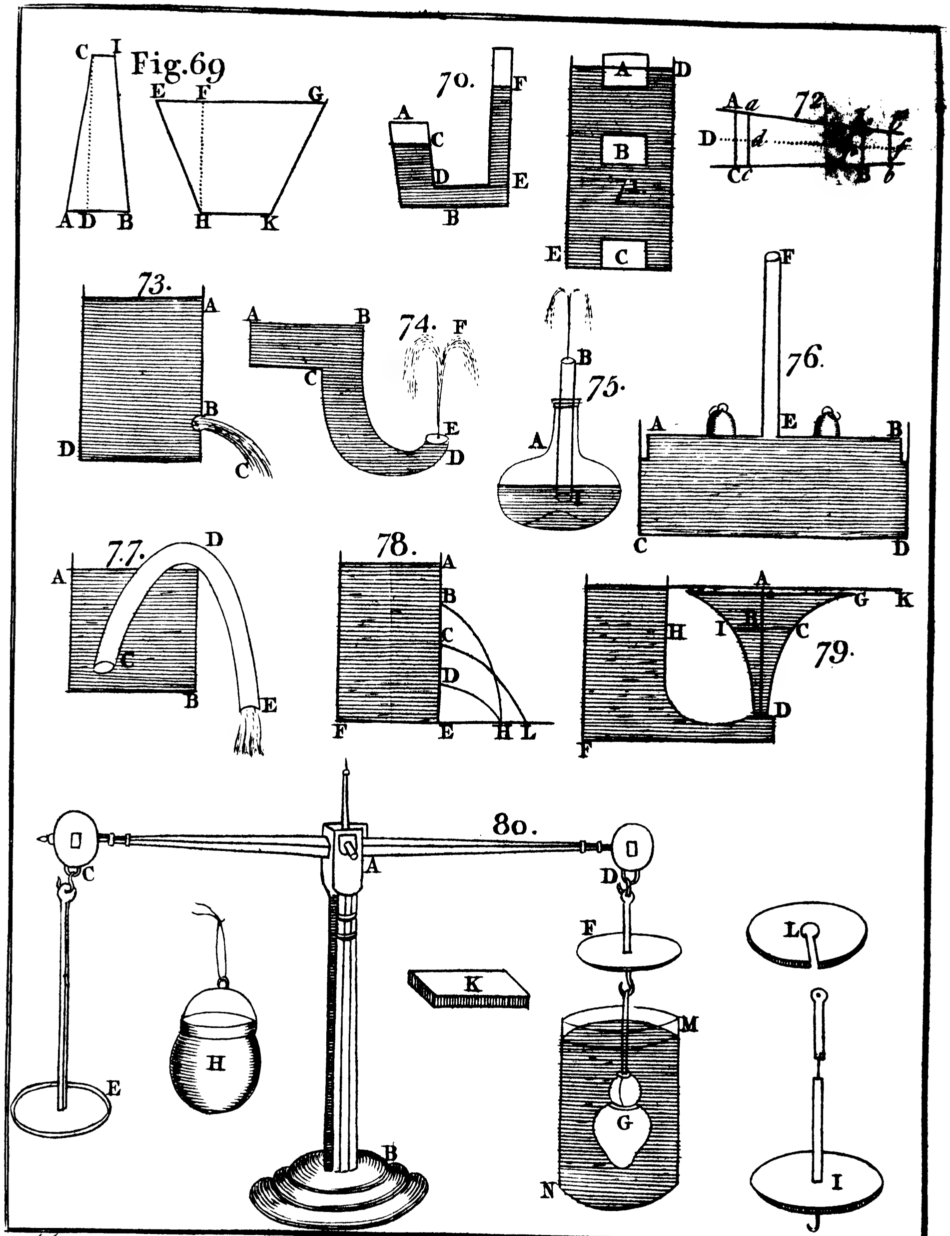
 bubble,

Fig. 80. bubble, when contrary. Whence the fluid bubble $= K \pm P$, for a heavier or lighter fluid. And the ſpecific gravities being as the weights of theſe equal bubbles; ſpecific gravity of water : ſpecific gravity of the fluid : : $K : K \pm P : : 1 : \frac{K+P}{K}$ the ſpecific gravity of the fluid. Where if P be o, it is the ſame as that of water.

PROP. LXXIII.

The air is a heavy body, and gravitates on all parts of the ſurface of the earth.

That the air is a fluid is very plain, as it yields to any the leaſt force that is impreſſed upon it, without making any ſenſible reſiſtance. But if it be moved briſkly, by ſome very thin and light body, as a fan, or by a pair of bellows, we become very ſenſible of its motion againſt our hands or face, and likewiſe by its impelling or blowing away any light bodies, that lie in the way of its motion. Therefore the air being capable of moving other bodies by its impulſe, muſt it ſelf be a body; and muſt therefore be heavy like all other bodies, in proportion to the matter it contains; and will conſequently preſs upon all bodies placed under it. And being a fluid, it will dilate and ſpread itſelf all over upon the earth: and like other fluids will gravitate upon, and preſs every where upon its ſurface. The gravity and preſſure of the air is alſo evident from experiments. For (fig. 70.) if water, &c. be put into the tube ABF, and the air be drawn out of the end F by an air-pump, the water will aſcend in the end F, and deſcend in the end A, by reaſon of the preſſure at A, which was taken off or diminiſhed at F. There are numberleſs experiments of th's ſort. And tho' theſe properties



Mechanics

Pl.VIII. pa.134.

perties and effects are certain, yet the air is a fluid so very fine and subtle, as to be perfectly transparent, and quite invisible to the eye. Fig. 70.

Cor. 1. *The air, like other fluids, will, by its weight and fluidity, insinuate itself into all the cavities, and corners within the earth; and there press with so much greater force, as the places are deeper.*

Cor. 2. *Hence the atmosphere, or the whole body of air surrounding the earth, gravitates upon the surfaces of all other bodies, whether solid or fluid, and that with a force proportional to its weight or quantity of matter.*

For this property it must have in common with all other fluids.

Cor. 3. *Hence the pressure, at any depth of water, or other fluid, will be equal to the pressure of the fluid together with the pressure of the atmosphere.*

Cor. 4. *Likewise all bodies, near the surface of the earth, lose so much of their weight, as the same bulk of so much air weight. And consequently, they are something lighter than they would be in a vacuum. But being so very small it is commonly neglected; tho' in strictness, the true or absolute weight is the weight in vacuo.*

PROP. LXXIV.

The air is an elastic fluid, or such a one, as is capable of being condensed or expanded. And it observes this law, that its density is proportional to the force that compresses it.

These properties of the air, are proved by experiments, of which there are innumerable. If you take a syringe, and thrust the handle inwards, you'll feel the included air act strongly against your

 hand;

Fig. hand; and the more you thruſt, the further the piſton goes in, but the more it reſiſts; and taking away your hand, the handle returns back to where it was at firſt. This proves its elaſticity, and alſo that air may be driven into a leſs ſpace, and condenſed.

75. Again, take a ſtrong bottle, and fill it half full of water, and cement a pipe BI, cloſe in it, going near the bottom; then inject air into the bottle thro' the pipe BI. Then the water will ſpout out at B, and form a jet; which proves, that the air is firſt condenſed, and then by its ſpring drives out the water, till it become of the ſame denſity as at firſt, and then the ſpouting ceaſes.

81. Likewiſe if a veſſel of glaſs AB be filled with water in the veſſel CD, and then drawn up with the bottom upwards; if any air is left in the top at A, the higher you pull it up, the more it expands; and the further the glaſs is thruſt down into the veſſel CD, the more the air is condenſed.

82. Again, take a crooked glaſs tube ABD open at the end A, and cloſe at D; pour in mercury to the hight BC, but no higher, and then the air in DC is in the ſame ſtate as the external air. Then pour in more mercury at A, and obſerve where it riſes to in both legs, as to G and H. Then you may always ſee that the higher the mercury is in the leg BH, the leſs the ſpace GD is, into which the air is driven. And if the hight of the mercury FH be ſuch as to equal the preſſure of the atmoſphere, then DG will be half DC; if it be twice the preſſure of the atmoſphere, DG will be $\frac{1}{3}$ DC, &c. So that the denſity is always as the weight or compreſſion. And here the part CD is ſuppoſed to be cylindrical.

Cor. 1. *The ſpace that any quantity of air takes up, is reciprocally as the force that compreſſes it.*

Cor.

Cor. 2. *All the air near the earth is in a state of compression, by the weight of the incumbent atmosphere.* Fig.

Cor. 3. *The air is denser near the earth, or at the foot of a mountain, than at the top of it, and in high places. And the higher from the earth the more rare it is.*

Cor. 4. *The spring or elasticity of the air is equal to the weight of the atmosphere above it; and produces the same effects.*

For they always balance and sustain each other.

Cor. 5. *Hence if the density of the air be increased; its spring or elasticity will likewise be increased in the same proportion.*

Cor. 6. *From the gravity and pressure of the atmosphere, upon the surfaces of fluids, the fluids are made to rise in any pipes or vessels, when the pressure within is taken off.*

PROP. LXXV.

The expansion and elasticity of the air is increased by heat, and decreased by cold. Or heat expands, and cold condenses the air.

This is also matter of experience; for tie a bladder very close with some air in it, and lay it before the fire, and it will visibly distend the bladder; and burst it if the heat is continued, and encreased high enough.

If a glass vessel AB (Fig. 81.) with water in it, 8
be turned upside down, with a little air in the top A; and be placed in a vessel of water, and hung over the fire, and any weight laid upon it to keep it down; as the water warms, the air in the top A, will by degrees expand, till it fills the glass, and by

Fig. 81. by its elaſtic force, drive all the water out of the glaſs, and a good part of the air will follow, by continuing the veſſel there. Many more experiments may be produced proving the ſame thing.

PROP. LXXVI.

The air will preſs upon the ſurfaces of all fluids, with any force; without paſſing thro' them, or entering into them.

If this was not ſo, no machine, whoſe uſe or action depends upon the preſſure of the atmoſphere, could do its buſineſs. Thus the weight of the atmoſphere preſſes upon the ſurface of water, and forces it up into the barrel of a pump, without any air getting in, which would ſpoil its working. Likewiſe the preſſure of the atmoſphere keeps mercury ſuſpended at ſuch a hight, that its weight is equal to that preſſure; and yet it never forces itſelf thro' the mercury into the vacuum above, though it ſtand never ſo long. And whatever be the texture or conſtitution of that ſubtle inviſible fluid we call air, yet it is never found to paſs through any fluid, tho' it be made to preſs never ſo ſtrongly upon it. For tho' there be ſome air incloſed in the pores of almoſt all bodies, whether ſolid or fluid; yet the particles of air cannot by any force be made to paſs thro' the body of any fluid; or forced through the pores of it, although that force or preſſure be continued never ſo long. And this ſeems to argue that the particles of air are greater than the particles or pores of other fluids; or at leaſt are of a ſtructure quite different from any of them.

PROP.

PROP. LXXVII.

The weight or pressure of the atmosphere, upon any base at the earth's surface; is equal to the weight of a column of mercury of the same base, and whose hight is from 28 to 31 inches, seldom more or less.

This is evident from the barometer, an instrument which shews the pressure of the air; which at some seasons stands at a hight of 28 inches, sometimes at 29, and 30, or 31. The reason of this is not, because there is at some times more air in the atmosphere, than at others; but because the air being an extremely subtle and elastic fluid, capable of being moved by any impressions, and many miles high; it is much disturbed by winds, and by heat and cold; and being often in a tumultuous agitation; it happens to be accumulated in some places, and consequently depressed in others; by which means it becomes denser and heavier where it is higher, so as to raise the column of mercury to 30 or 31 inches. And where it is lower, it is rarer and lighter, so as only to raise it to 28 or 29 inches. And experience shews, that it seldom goes without the limits of 28 and 31.

Cor. 1. *The air in the same place does not always continue of the same weight; but is sometimes heavier, and sometimes lighter; but the mean weight of the atmosphere, is that when the quicksilver stands at about 29 $\frac{1}{2}$ inches.*

Cor. 2. *Hence the pressure of the atmosphere upon a square inch at the earth's surface, at a medium, is very near 15 pounds, averdupoise.*

For an inch of quicksilver weighs 8.102 ounces.

Cor. 3. *Hence also the weight or pressure of the atmosphere, in its lightest and heaviest state, is equal to the*

Fig. *the weight of a column of water, 32 or 36 feet high; or at a medium 34 feet.*

For water and quicksilver are in weight nearly as 1 to 14.

Cor. 4. *If the air was of the same density to the top of the atmosphere, as it is at the earth; its hight would be about 5 $\frac{1}{3}$ miles at a medium.*

For the weight of air and water are nearly as 12 to 1000.

Cor. 5. *The density of the air in two places distant from each other but a few miles, on the earth's surface and in the same level; may be looked on to be the same, at the same time.*

Cor. 6. *The density of the air at two different altitudes in the same place, differing only by a few feet; may be looked on as the same.*

Cor. 7. *If the perpendicular hight of the top of a syphon from the water, be more than 34 feet, at a mean density of the air. The syphon cannot be made to run.*

For the weight of the water in the legs will be greater than the pressure of the atmosphere, and both columns will run down, till they be 34 feet high.

Cor. 8. *Hence also the quicksilver rises higher in the barometer, at the bottom of a mountain than at the top. And at the bottom of a coal pit, than at the top of it.*

Fig.

SCHOLIUM.

Hence the denſity of the air may be found at any hight from the earth, as in the following table.

Miles	denſity	Miles	denſity
$\frac{1}{4}$	.9564	10	.1700
$\frac{1}{2}$	.9146	20	.02917
$\frac{3}{4}$	.8748	30	.005048
1	.8372	40	.000881
2	.7012	50	.000155
3	.5871	100	.0000000298
4	.4917		
5	.4119		

The firſt and third columns are the hight in miles from the ſurface of the earth. And the ſecond and fourth columns, ſhew the denſity at that hight; ſuppoſing the denſity at the ſurface of the earth, to be 1.

The denſity at any hight is eaſily calculated by this ſeries. Put r = radius of the earth, h = hight from the ſurface, both in feet. Then the denſity at the hight h, is the number belonging to the logarithm, denoted by this ſeries $-\frac{h}{68444}-\frac{h}{r}A-\frac{h}{r}B-\frac{h}{r}C$ &c. where A, B, C, &c. are the preceding terms. The terms here will be alternately negative and affirmative. But the firſt term alone is ſufficient when the hight is but a few miles.

By the weight and preſſure of the atmoſphere, the operations of pneumatic engines may be accounted for and explained. I ſhall juſt mention one or two.

A PUMP.

Fig. 83. is a *common pump*. AB the barrel or body of the pump, being a hollow cylinder, made of 83.

Fig. 83. of wood or lead. CD the handle movable about the pin E. DF an iron rod moving about a pin D; this rod is hooked to the bucket or ſucker FG, which moves up and down within the pump. The bucket FG is hollow, and has a valve or clack L at the top opening upwards. H a plug fixed at the bottom of the barrel, being likewiſe hollow, and a valve at I opening alſo upwards. BK the bottom going into the well at K; the pipe below B need not be large, being only to convey the water out of the well into the body of the pump. The plug H muſt be fixed cloſe that no water can get between it and the barrel; and the ſucker FG, is to be armed with leather, to fit cloſe that no air or water can get thro' between it and the barrel.

When the pump is firſt wrought, or any time in dry weather when the water above the ſucker is waſted, it muſt be primed, by pouring in ſome water at the top A to cover the ſucker, that no air get through. Then raiſing the end C of the handle, the bucket F deſcends, and the water will riſe thro' the hollow GL, preſſing open the valve L. Then putting down the end C raiſes the bucket F, and the valve L ſhuts by the weight of the water above it. And at the ſame time the preſſure of the atmoſphere forces the water up thro' the pipe KB, and opening the valve I, it paſſes thro' the plug into the body of the pump. And when the ſucker G deſcends again, the valve I ſhuts, and the water cannot return, but opening the valve L, paſſes thro' the ſucker GL. And when the ſucker is raiſed again, the valve L ſhuts again, and the water is raiſed in the pump. So that by the motion of the piſton up and down, and the alternate opening and ſhutting of the two valves; water is continually raiſed into the body of the pump, and diſcharged at the ſpout M.

The

The distance KG, from the well to the bucket, Fig. must not be above 32 feet; for the pressure of the 83. atmosphere will raise the water no higher, and if it is more, the pump will not work. It is evident a pump will work better when the atmosphere is heavy than when it is light, there being a twelfth or fifteenth part difference, at different times. And when it is lightest it is only equal to 32 feet. Wherefore the plug H must always be placed so low, as that the sucker GL may be within that compass.

A BAROMETER.

Fig. 84. is a *Barometer*, or an instrument to mea- 84. sure the weight of the air. It consists of a glass cone ABC hollow within, filled full of mercury, and hermetically sealed at the end C, so that no air be left in it. When it is set upright, the mercury descends down the tube BC, into the bubble A, which has a little opening at the top A, that the air may have free ingress and egress. At the top of the tube C, there must be a perfect vacuum. This is fixed in a frame, and hung perpendicular against a wall. Near the top C, on the frame, is placed a scale of inches, shewing how high the mercury is in the tube BC, above the level of it in the bubble A, which is generally from 28 to 31 inches, but mostly about 29 or 30. Along with the scale of inches, there is also placed a scale of such weather as has been observed to answer the several hights of the quicksilver. Such a scale you have annexed to the 84th figure. In dividing the scale of inches, care must be taken to make proper allowance for the rising or falling of the quicksilver in the bubble A, which ought to be about half full, when it stands at $29\frac{1}{2}$, which is the mean hight. For whilst the quicksilver rises an inch at C, it descends a little in the bubble A, and that descent

Fig. 84. descent must be deducted, which makes the divisions be something less than an inch. These inches must be divided into tenth parts, for the more exact measuring the weight of the atmosphere. For the pillar of mercury in the tube is always equal to the weight of a pillar of the atmosphere of the same thickness. And as the hight of the quicksilver increases or decreases, the weight of the air increases or decreases accordingly. The tube must be near 3 feet long, and the bore not less than $\frac{1}{5}$ or $\frac{1}{6}$ of an inch, in diameter, or else the quicksilver will not move freely in it.

By help of the barometer, the hight of mountains may be measured by the following table. In which the first column is the hight of the mountain, &c. in feet or miles; the second the hight of the quicksilver; and the third the descent of the quicksilver in the barometer; and this at a mean density of the air.

Feet	High Barom.	Descent	Feet	High Barom.	Descent.
0	29.500				
100	29.400	.100	2600	27.028	2.472
200	29.301	.199	2700	26.938	2.562
300	29.203	.297	2800	26.848	2.652
400	29.105	.395	2900	26.758	2.742
500	29.007	.493	3000	26.668	2.832
600	28.910	.590	3100	26.578	2.922
700	28.812	.688	3200	26.489	3.011
800	28.716	.784	3300	26.400	3.100
900	28.619	.881	3400	26.311	3.189
1000	28.523	.977	3500	26.222	3.278
1100	28.428	1.072	3600	26.136	3.364
1200	28.332	1.168	3700	26.049	3.451
1300	28.237	1.263	3800	25.961	3.539
1400	28.143	1.357	3900	25.874	3.626
1500	28.048	1.452	4000	25.786	3.714
1600	27.954	1.546	4100	25.699	3.801
1700	27.860	1.640	4200	25.613	3.887
1800	27.766	1.734	4300	25.527	3.973
1900	27.672	1.828	4400	25.441	4.059
2000	27.579	1.921	4500	25.355	4.145
2100	27.487	2.013	4600	25.270	4.230
2200	27.394	2.106	4700	25.185	4.315
2300	27.302	2.198	4800	25.101	4.399
2400	27.210	2.290	4900	25.017	4.483
2500	27.119	2.381	5000	24.933	4.567

Fig. 84.

The Table continued in MILES.

Miles	H. Barom.	Deſcent	Miles	H. Barom.	Deſcent
0.	29.50				
0.25	28.21	1.29	3.25	16.57	12.93
0.50	26.98	2.52	3.50	15.85	13.65
0.75	25.80	3.70	3.75	15.16	14.34
1.	24.70	4.80	4.	14.50	15.00
1.25	23.62	5.88	4.25	13.87	15.63
1.50	22.60	6.90	4.50	13.27	16.23
1.75	21.62	7.88	4.75	12.70	16.80
2.	20.68	8.82	5.	12.15	17.35
2.25	19.78	9.72	5.25	11.62	17.88
2.50	18.93	10.57	5.50	11.12	18.38
2.75	18.11	11.39	5.75	10.64	18.86
3.	17.32	12.18	6.	10.18	19.32

This table is made from a table of the air's denſity, made as in Schol. Prop. LXXVII. And then multiplying all the numbers thereof by 29.5 the mean denſity of the air. For the denſity of the air at any height above the earth is as the weight of the atmoſphere above it, (by Prop. LXXIV.); and that is as the height of the mercury in the barometer.

A WATER BAROMETER.

85. A barometer may alſo be made of water as in fig. 85, which is a water barometer. AB is a glaſs tube open at both ends, and cemented cloſe in the mouth of the bottle EF, and reaching very near the bottom. Then warming the bottle at the fire, part of the air will fly out; then the end A is put into a veſſel of water mixed with cochineal, which will go thro' the pipe into the bottle as it grows cold. Then it is ſet upright; and the water may

may be made to ſtand at any point C, by ſucking Fig.
or blowing at A. And if this barometer be kept 85.
to the ſame degree of heat, by putting it in a veſſel of ſand, it will be very correct for taking ſmall altitudes; for a little alteration in the weight of the atmoſphere, will make the water at C riſe or fall in the tube very ſenſibly. But if it be ſuffered to grow warmer, the water will riſe too high in the tube, and ſpoil the uſe of it; ſo that it muſt be kept to the ſame temper.

If a barometer was to be made of water put into an exhauſted tube, after the manner of quickſilver; it would require a tube 36 feet long or more; which could hardly find room within doors. But then it would go 14 times more exact than quickſilver; becauſe for every inch the quickſilver riſes, the water would riſe 14; from whence every minute change in the atmoſphere would be diſcernable.

And the water barometer above deſcribed will ſhew the variation of the air's gravity as minutely as the other, if the bottle be large to hold a great quantity of air. And in any caſe, by reducing the bottle (ſo far as the air is contained) to a cylinder; and put D = diameter of the bottle, d = diameter of the pipe, p = height of air, x = riſing in the pipe, all in inches. Then the height of a hill in feet will be nearly $\overline{1 + \frac{408dd}{pDD}} \times 71x$. And if y = height of the hill or any aſcent, Q = $\frac{408dd}{pDD}$. Then $x = \frac{y}{\overline{1 + Q} \times 71}$ very near, at a mean denſity of the air.

A THERMOMETER.

Fig. 86. is a *thermometer*, or an inſtrument to 86.
meaſure the degrees of heat and cold. AB is a hollow

Fig. 86. hollow tube near two foot long, with a ball at the bottom; it is filled with ſpirits of wine mixed with cochineal, half way up the neck; which done, it is heated very much, till the liquor fill the tube, and then it is ſealed hermetically at the end A. Then the ſpirit contracts within the tube as it cools. It is incloſed in a frame, which is graduated into degrees, for heat and cold. For hot weather dilates the ſpirit, and makes it run further up the tube; and cold weather on the contrary, contracts it, and makes it ſink lower in the tube. And the particular diviſions, ſhew the ſeveral degrees of heat and cold; againſt the principal of which, the words heat, cold, temperate, &c. are written.

They that would ſee more machines deſcribed, may conſult my large book of Mechanics, where he will meet with great variety.

FINIS.

ERRATUM.

Page 35, Line 10 from the bottom, *read*, Cor. 1. Hence

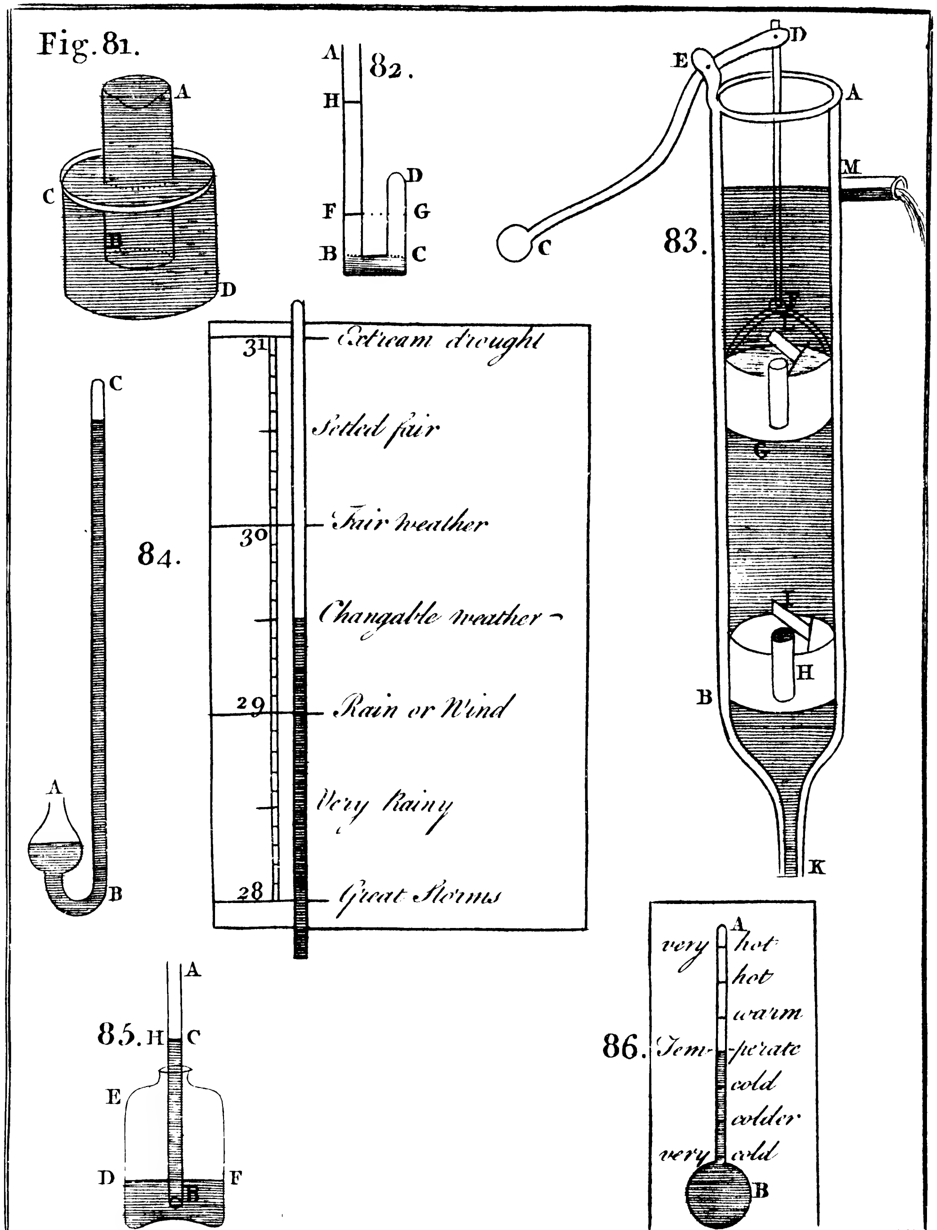


Fig. 81.
A
B
C
D
82.
A
H
F
B
D
G
C
83.
D
E
A
M
C
G
H
B
K
84.
C
A
B
31
30
29
28
Extream drought
Setled fair
Fair weather
Changable weather
Rain or Wind
Very Rainy
Great Storms
85.
H
C
A
E
D
F
B
86.
A
very hot
hot
warm
Tem-perate
cold
colder
very cold
B

THE PROJECTION OF THE SPHERE,

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PRINCIPLES,

And explaining the

PRACTICE

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M DCC LXIX.

THE

PREFACE.

THE Projection of the Sphere, or of its Circles, has the ſame relation to Spherical Trigonometry, that practical Geometry has to plane Trigonometry. For as the one ſaves a deal of Calculation, by drawing a few right Lines, ſo does the other by drawing a few Circles. The Projection of the Sphere gives a Learner a good Idea of the Sphere and all its Circles, and of their ſeveral Poſitions to one another, and conſequently of Spherical Triangles, and the Nature of Spherical Trigonometry.

I have here delivered the Principles of three ſorts of Projection, in a ſmall compaſs; and yet the Reader will find here, all that is eſſential to the ſubject; and yet nothing ſuperfluous; for I think no more need be ſaid, or indeed can be ſaid about it, to make it intelligible and practicable. For here is laid down, not only the whole Theory, but the Practice likewiſe. Yet the practical Part is entirely diſengaged from the Theory; ſo that any body (tho' he has no deſire or leiſure to attain to the Theory,) may neverthele'ſs, by help of the Problems, make himſelf Maſter of the Practice. For which end I have endeavoured to make all the rules relating to practice, plain, ſhort, and eaſy, and at the ſame time full and clear.

It is true the ſolution of Problems this way, muſt be allowed to be imperfect; for there will always be ſome errors in working, as well as in the inſtruments *we work with. But nobody in seeking an accurate solution to a Problem, will trust to a Projection by scale and compass; because this cannot be depended on in cases of great nicety. Yet where no great exactness is required it will be found very ready and useful; and, besides, will serve to prove and confirm the solution obtaind by* Calculation.

But then this defect is abundantly recompensed by the easiness of this method. For by scale and compass only, all sorts of Problems belonging to the Sphere, as in Astronomy, Geography, Dialling, &c. may be solved with very little trouble, which require a great deal of time and pains, to work out trigonometrically by the tables. It likewise affords a great pleasure to the mind, that one can, in a little time, describe the whole furniture of Heaven, and Earth, and represent them to the eye, in a small scheme of paper.

But its principal use is for such persons (and that is by far the greater number) as having no opportunity for learning Spherical Trigonometry, have yet a desire to resolve some Problems of the Sphere. For such as these, this small Treatise will be of particular service, because the practical rules, especially of any one sort of Projection, may be learned in a very little time, and are easily remembered. So that I have some hopes I shall please all my Readers, whether theoretical or practical.

W. E.

THE

THE PROJECTION OF THE SPHERE IN PLANO.

DEFINITIONS.

1. *PROJECTION* of the sphere is the representing its surface upon a plane, called the *Plane of Projection.*

2. *Orthographic* Projection, is the drawing the circles of the sphere upon the plane of some great circle, by lines perpendicular to that plane, let fall from all the points of the circles to be projected.

3. The *Stereographic* Projection, is the drawing the circles of the sphere upon the plane of one of its great circles, by lines drawn from the pole of that great circle to all the points of the circles to be projected.

4. The *Gnomonical* Projection, is the drawing the circles of an hemisphere, upon a plane touching it in the vertex, by lines or rays issuing from the center of the hemisphere, to all the points of the circles to be projected.

5. The *Primitive* circle is that on whose plane the sphere is projected. And the pole of this circle is called the *Pole* of Projection. The point from whence the *projecting* right lines issue is the *projecting Point.*

6. The *Line of Meaſures* of any circle is the common interſection of the plane of projection, and another plane that paſſes thro' the eye, and is perpendicular both to the plane of projection, and to the plane of that circle.

SCHOLIUM.

There are other Projections of the Sphere, as the *Cylindrical*, the *Scenographic* which belongs to Perſpective, the *Globical* which belongs to Geography, *Mercators*, for which ſee Navigation, &c.

AXIOM.

The *Place* of any viſible point of the Sphere upon the plane of projection, is where the projecting line cuts that plane.

Cor. *If the eye be applied to the projecting point, it will view all the circles of the Sphere, and every part of them, in the projection, juſt as they appear from thence in the Sphere itſelf.*

SCHOLIUM.

The Projection of the Sphere is only the ſhadow of the circles of the Sphere upon the plane of Projection, the light being in the place of the eye or projecting point.

The Signification of ſome Characters.

$+$ added to.
$-$ ſubtracting the following quantity.
$\angle$ an angle.
$=$ equal to.
$\perp$ perpendicular to.
$\parallel$ parallel to.
$::$ a proportion.

SECT.

SECT. I.

The Orthographic Projection of the SPHERE.

PROP. I.

IF a right line AB *is projected upon a plane, it is projected into a right line; and its length will be to the length of the projection, as radius to the cosine of its inclination above that plane.* Fig. 3.

For let fall the perpendiculars A*a*, B*b* upon the plane of projection, then *ab* will be the line, it is projected into; but by trigonometry AB : is to A*o* or *ab* : : as radius : to the sine of B or cosine of *o*AB.

Cor. 1. *If a right line is projected upon a plane, parallel thereto, it is projected into a right line parallel and equal to itself.*

Cor. 2. *If an angle be projected upon a plane which is parallel to the two lines forming the angle; it is projected into an angle equal to itself.*

Cor. 3. *Any plain figure projected upon a plane parallel to itself, is projected into a figure similar and equal to itself.*

Cor. 4. *Hence also the area of any plain figure, is to the area of its projection : : as radius, to the cosine of its elevation or inclination.*

PROP. II.

A circle perpendicular to the plane of projection, is projected into a right line equal to its diameter.

For projecting lines drawn through all the points of the circle fall in the common section of the planes

Fig. of the circle and of projection, which is a right line (Geom. V. 3.), and equal to the diameter of the circle; because the planes intersect in that diameter. *Q. E. D.*

Cor. *Hence any plane figure, perpendicular to the plane of projection is projected into a right line. For the perpendiculars from every point, will all fall in the common intersection of the figure with the plane of projection.*

PROP. III.

1. *A circle parallel to the plane of projection is projected into a circle equal to itself, and concentric with the primitive.*

Let BOD be the circle, I its center, C the center of the sphere, the points I, B, O, D, are projected into the points C, L, F, G. And therefore OICF, and BICL are rectangled parallelograms. Consequently LC = BI = OI = FC, (Geom. III. 1.). *Q. E. D.*

Cor. *The radius* CL *or* CF *is the cosine of the circle's distance from the primitive, for it is the sine of* AB.

PROP. IV.

2. *An inclined circle is projected into an ellipsis whose transverse axis is the diameter of the circle.*

Let ADBH be the inclined circle, P its center; and let it be projected into *adbh*; draw the plane ABFC*a* through the center C of the sphere, perpendicular to the plane of the given circle and plane of projection, to intersect them in the lines AB, *ab*; draw GPH, DE, perpendicular, and DQ parallel to AB; then because the line GP, and the plane of projection are both perpendicular to the

the plane ABF; therefore GH is parallel to the Fig.
plane of projection, and therefore to gh.

In the circle ADB, $DQ^2 = GQH = gqh$, and and $BP^2 = GP^2 = gp^2$. And (Geom. V. 12.) BP : EP or DQ : : $bp : ep$ or dq, and $BP^2 : DQ^2 :: bp^2 : dq^2$; that is, $gp^2 : gqh :: bp^2 : dq^2$; and therefore $agbh$ is an ellipsis, whose transverse gh is the diameter of the circle. *Q. E. D.*

Cor. 1. *Since ab is perpendicular to gh, therefore ab is the conjugate axis; and is twice the sine of the* < ABb *to the radius gp; that is, the conjugate axis is equal to twice the cosine of the inclination, to the radius of the circle.*

Cor. 2. *The transverse axis is equal to twice the cosine of its distance from its parallel great circle. For* gh = GH = 2AP = *twice the sine of* AK.

Cor. 3. *The extremities of the conjugate axis are distant from the center of the primitive, by the sines of the circles nearest and greatest distance from the pole of the primitive. Thus a*C *is the sine of* AN, *and b*C *the sine of* BN.

Cor. 4. *Hence also it is plain that the conjugate axis always passes thro' the center* C *of the primitive; and is always in the line of measures of that circle.*

SCHOLIUM.

Every circle in the projection represents two equal 3.
circles, parallel and equidistant from the primitive. Every right line represents two semicircles, one towards the eye, the other in the opposite side. Every ellipsis represents two equal circles, but contrarily inclined as AB, CD; one above the primitive the other below it.

And now the Theory being laid down, it remains only to deduce thence, some short rules for practice, as follows.

PROP.

Fig.

PROP. V. *Prob.*

5. *To project a circle parallel to the primitive.*

Rule.

Take the complement of its diſtance from the primitive, and ſet it from A to E; and with the center C and radius CD = perpendicular EF, deſcribe the circle DgG.

By the plain ſcale.

Take the ſine of its diſtance from the pole of the primitive; with that radius and the center C deſcribe the circle.

PROP. VI. *Prob.*

4. *To project a right circle, or one that is perpendicular to the plain of projection.*

Rule.

Thro' the center C of the primitive, draw the diameter AB, and take the diſtance from its parallel great circle, and ſet from A to E, and from B to D, and draw ED, the right circle required.

By the ſcale.

Take the ſine of the circle's diſtance from its parallel great circle AB, and at that diſtance draw a parallel ED for the circle required.

PROP. VII. *Prob.*

To project a given oblique circle.

Rule.

6. Draw the line of meaſures AB, and take the circle's neareſt diſtance from the primitive, and ſet from

from B to D, upwards if it be above the primitive; Fig.
or downward, if below; likewiſe take its greateſt 6.
diſtance, and ſet from A to E, and draw ED, and let fall the perpendiculars EF, DG; and biſect FG in H, and erect the perpendicular KHI, making KH = HI = half ED; then deſcribe an ellipſis (by the Conic Sections) whoſe tranſverſe is IK and conjugate FG; and that ſhall repreſent the circle given.

By the ſcale.

Draw the line of meaſures AB; and take the 6.
ſines of the circle's neareſt and greateſt diſtance from the pole of the primitive, and ſet them from the center C to F and G, (both ways if the circle encompaſs the pole, but the ſame way if it lie on one ſide the pole;) biſect FG in H, and erect HK, HI perpendicular to FG, and = to the radius of the circle given, or the ſine of its diſtance from its own pole; about the axes FG, KI deſcribe an ellipſis, and it is done.

S C H O L I U M.

An ellipſis great or ſmall may be deſcribed by 10.
points, thus; thro' the center D of the circle and ellipſis, draw BD ⊥ the tranſverſe AR; and on AR erect a ſufficient number of perpendiculars IK, *ik* &c. and make as DB or DA : DE : : IK : IF : : *ik* : *if* &c. then thro' all the points E, F, *f*, &c. draw a curve. See Prop. 76. ellipſis.

P R O P. VIII. *Prob.*

To find the pole of a given ellipſis.

Rule.

Thro' the center of the primitive C, draw the 7.
conjugate of the ellipſis; on the extreme points F, G, erect the perpendiculars FE, GD, or ſet the tranſverſe

Fig. transverse IK from E to D, and bisect ED in R,
7. and let fall RP perpendicular to AB, then is P the pole.

By the scale.

7. Take CF, and CG, and apply to the sines, and find the degrees answering or the supplements; then take the sine of half the sum of these degrees, if F, G be both on one side of C, or the sine of half the difference, if they lie on contrary sides; and set it from C to the pole P.

Or thus; apply the semi-transverse IH to the sines, and set the degrees from E to R; and draw RP ⊥ to AB; and P is the pole.

PROP. IX. *Prob.*

To measure an arch of a parallel circle; or to set any number of degrees on it.

Rule.

With the radius of the parallel, and one foot in C describe a circle Gg, draw CGB, and C*gb*; then B*b* will measure the given arch G*g*; or G*g* will contain the given number of degrees set from B to *b*. So that either being given finds the other.

PROP. X. *Prob.*

To measure any part of a right circle.

Rule.

8. In the right circle ED, let EA = AD; and let AB be to be measured. Make CF = AE; with extent BA = FG describe the arch GI; draw CGK to touch it in G; then is HK the measure of AB. For FG = S. < HCK to the radius CF or AE, and BA is the same, by Cor. Prop. III.

Other-

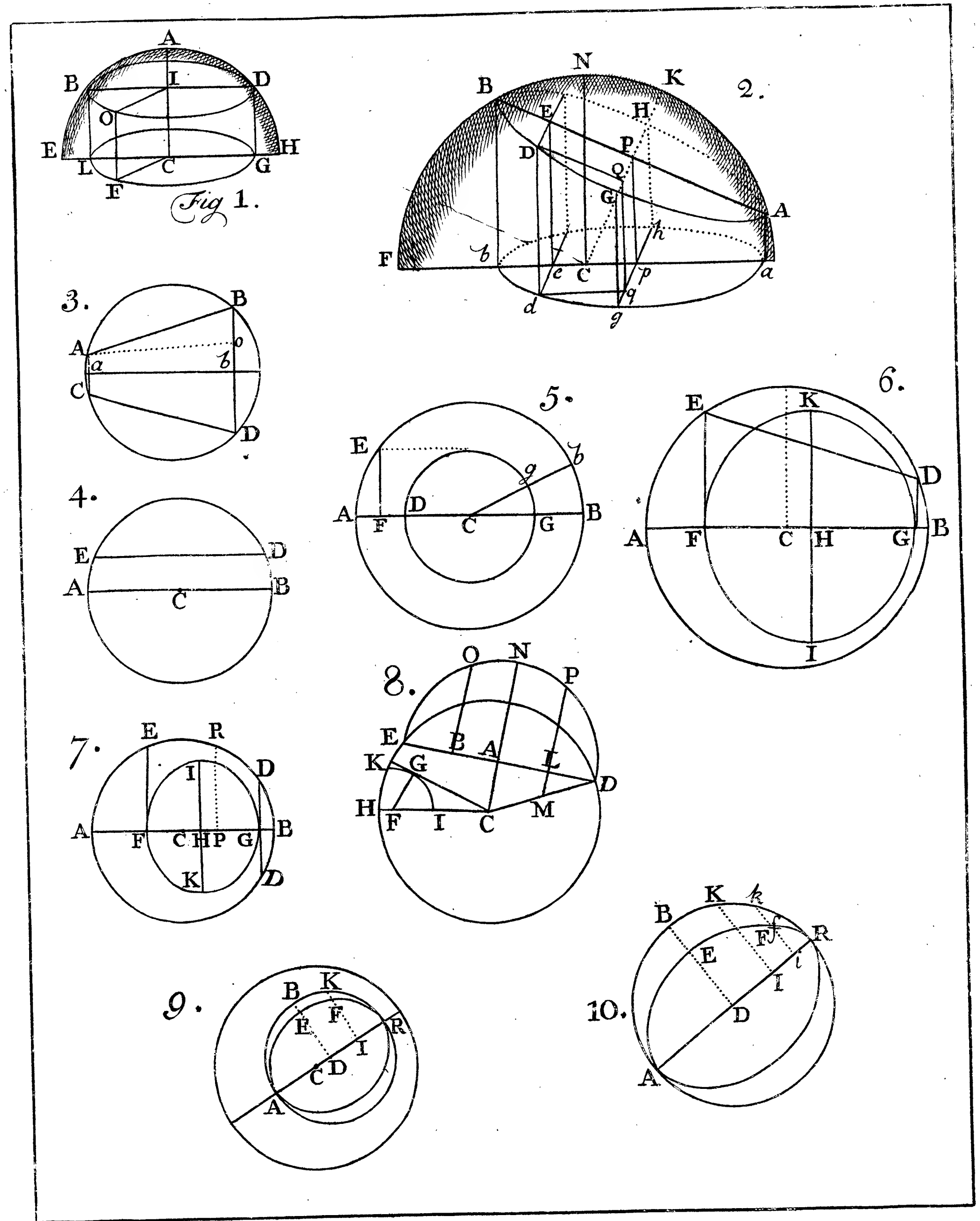


Fig 1.
2.
3.
4.
5.
6.
7.
8.
9.
10.

Fig. 8.

Otherwise thus.

On the diameter ED, describe the semicircle END, draw AN, BO, LP perpendicular to ED, then ON is the measure of BA, and NP of AL; and ON or NP may be measured as in Prop. IX.

By the scale.

Let AL be to be measured. Draw CD; and LM parallel to AC, then CM applied to the sines gives the degrees. For radius CD : AD : : CM : AL.

Cor. *If the right circle passes thro' the center, there is no more to do, but to raise perpendiculars on it, which will cut the primitive, as required. Or apply the part of the right circle to the line of sines.*

PROP. XI. *Prob.*

To set off any number of degrees upon a right circle, DE.

Rule.

Draw CA ⊥ DE, and make the ∠ HCK = the degrees given, make CF = radius AE, take FG the nearest distance, and set from A to B; then AB = ∠ HCK, the degrees proposed. 8.

Otherwise thus.

On ED describe the semicircle END, then by Prop. IX. set off NP = degrees given, draw PL perpendicular to ED, then AL contains the degrees required.

Or thus by the scale.

Draw CD, take the given degrees of the sines, and set from C to M, and draw ML parallel to CA, then AL = arch required.

PROP.

Fig.

PROP. XII. *Prob.*

9. *To measure an arch of an ellipsis; or to set any*
10. *number of degrees upon it.*

Rule.

About AR the transverse axis of the ellipsis, describe a circle ABR; erect the perpendiculars BED, KFI, on AR; then BK is the measure of EF, or EF is the representation of the arch BK. And BK is to be measured, or any degrees set upon it, as in Prop. IX.

SCHOLIUM.

These Problems are all evident from the three first propositions, and need no other demonstration. If the sphere be projected on any plane parallel to the primitive, the projection will be the very same; for being effected by parallel lines, which are always at the same distance, there will be produced the same figure, or representation. Of all orthographic projections, those on the meridian or on the solstitial colure, commonly called the Analemma, is most useful; because a great many of the circles of the sphere fall into right lines or circles, whereas in the projections upon other planes, they are projected into ellipses, which are hard to describe; which makes these sorts of projection to be neglected.

And by the same rules that the circles of the sphere are projected upon a plane, any other figure may likewise be orthographically projected; by letting fall perpendiculars upon the plane from all the angles, or all the points of the figure, and joining these points with right or curve lines, as they are in the figure itself.

By this kind of projection, either the convex or concave side of the sphere, may be projected;

which

which is peculiar to this ſort of projection; that Fig. 9. is, either the hemiſphere towards you, or that from you, may be projected upon the plane of its great circle. And ſince in ſome caſes they both have the ſame appearance, it ought to be mentioned whether it is.

But if both the convex and concave ſides of the *ſame hemiſphere* be projected; that is, if you make two projections, one for the convex, the other for the concave ſide; the circles in one will be inverted in reſpect of the other, the right to the left, &c. Becauſe in looking at the ſame hemiſphere, it will not have the ſame appearance, when you look at the contrary ſides of it; becauſe you look contrary ways at it, to ſee the external and internal ſurfaces.

ig.

SECT. II.

The Stereographic Projection of the SPHERE.

PROP. I.

ANY circle passing thro' the projecting point, is projected into a right line.

For all lines drawn from the projecting point, to this circle, pass thro' the intersection of this circle and plane of projection, which is a right line.

Cor. 1. *A great circle passing thro' the poles of the primitive is projected into a right line passing thro' the center.*

Cor. 2. *Any circle passing thro' the projecting point is projected into a right line perpendicular to the line of measures, and distant from the center, the semitangent of its nearest distance from the pole opposite to*
12. *the projecting point. Thus the circle* AE *is projected into a right line passing thro'* G, *and perpendicular to* BC, *the line of measures, and* GC *is the semitangent of* EM.

PROP. II.

Every circle (that passes not through the projecting point) is projected into a circle.

11. *Case* I. Let the circle EF be parallel to the primitive BD; lines drawn to all points of it from the projecting point A, will form a conic surface, which being cut parallel to the base by the plane BD, the section GH (into which EF is projected) will be a circle by the conic sections.

Case

Case II. Let BH be the line of measures to the circle EF, draw FK parallel to BD, then arch AK = AF, and therefore < AFK or AHG = AEF; therefore in the triangles AEF, AGH, the angles at E and H are equal, and the angle A common; therefore the angles at F and G are equal. Therefore the cone of rays AEF (whose base EF is a circle) is cut by subcontrary section, by the plane of projection BD, and therefore, by the conic sections, the section GH (which is the projection of the circle EF) will also be a circle. *Q. E. D.* Fig. 12.

Cor. *When* AF *is equal to* AG, *the circle* EF *is projected into a circle equal to it self.*

For then the similar triangles AHG and AEF, will also be equal, and GH = EF.

PROP. III.

Any point on the sphere's surface is projected into a point, distant from the center, the semi-tangent of its distance from the pole opposite to the projecting point.

Thus the point E is projected into G, and F into H; and CG is the semi-tangent of EM, and CH of MF.

Cor. 1 *A great circle perpendicular to the primitive is projected into a line of semi-tangents passing thro' the center, and produced infinitely.*

For MF is projected into CH its semi-tangent, and EM into the semi-tangent CG.

Cor. 2. *Any arch* EM *of a great circle perp. to the primitive; is projected into the semi-tangent of it.*

Thus EM is projected into GC.

Cor. 3. *Any arch* EMF *of a great circle is projected into the sum of its semi-tangents, of its greatest*

Fig. 12. *and least distances from the opposite pole M, if it lye on both sides of M, or the dif. of the semi-tangents, when all on one side.*

PROP. IV.

13. *The angle made by two circles on the surface of the sphere is equal to that made by their representatives upon the plane of projection.*

Let the angle BPK be projected. Thro' the angular point P and the center C, draw the plane of a great circle PED perpendicular to the plane of projection EFG. Let a plane PHG touch the sphere in P; then since the circle EPD is perpendicular both to this plane and to the plane of projection, therefore it is perpendicular to their intersection GH. The angles made by circles are the same as those made by their tangents, therefore in the plane PGH, draw the tangents PH, PF, PG to the arches, PB, PD, PK; and these will be projected into the lines pH, pF, pK: Now I say the $< HPG = < HpG$. For the angle $CPF =$ a right angle $= CpA + CAP$; therefore taking away the equal angles CPA and CAP, and $<pPF = CpA$ or PpF; consequently $pF = PF$. Therefore in the right angled triangles PFG and pFG, there are two sides equal and the included $<$ right; therefore hypothenuse $PG = pG$. And for the same reason in the right angled triangles PFH and pFH, $PH = pH$. Lastly in the triangles PHG and pHG, all the sides are respectively equal, and therefore $< P = < p$. *Q. E. D.*

Cor. 1. *The rumb lines projected make the same angles with the meridians as upon the globe; and therefore are logarithmic spirals on the plain of the equinoctial. For every particle of the rumb coincides with some great circle.*

Cor.

Cor. 2. *The angle made by two circles on the sphere is equal to the angle made by the radii of their projections at the point of intersection. For the angle made by two circles on a plane is the same with that made by their radii drawn to the point of intersection.* Fig.

PROP. V.

The center of a projected (lesser) circle perpendicular to the primitive, is in the line of measures distant from the center of the primitive, the secant of the lesser circles distance from its own pole; and its radius is the tangent of that distance.

Let A be the projecting point, EF the circle to be projected, GH the projected diameter. From the centers C, D draw CF, DF, and the triangles CFI, DFI are right angled at I; then $\angle$ IFC $=$ $\angle$ FCA $=$ 2$\angle$ FEA or 2FEG $\doteq$ 2 $\angle$FHG $=$ $\angle$ FDG, therefore IFC $+$ IFD $=$ FDG $+$ IFD $=$ a right angle; that is CFD is a right angle, and the line CD is the secant of BF, and the radius FD is the tangent of it. *Q. E. D.* 14.

Cor. *If these circles be actually described, 'tis plain the radius* FD *is a tangent to the primitive at* F, *where the lesser circle cuts it.*

PROP. VI.

The center of Projection of a great circle is in the line of measures, distant from the center of the primitive, the tangent of its inclination to the primitive; and its radius is the secant of its inclination.

Let A be the projecting point, EF the great circle, GH the projected diameter, D the center; 15.

Fig. 15. draw DA. The angle EAF being in a semicircle is right. In the right angled triangle GAH, AC is perpendicular to GH, therefore < GAC = AHC and their double, ECB = ADC, and their complements. ECF = CAD. Therefore CD is the tangent of ECI, and radius AD its secant. *Q. E. D.*

16. Cor. 1. *If the great oblique circle* AGBH *be actually described upon the primitive* AIB. *I say, all great circles passing thro'* G *will have the centers of their projections in the line* RS *drawn thro' the center* D, *perpendicular to the line of measures* IH.

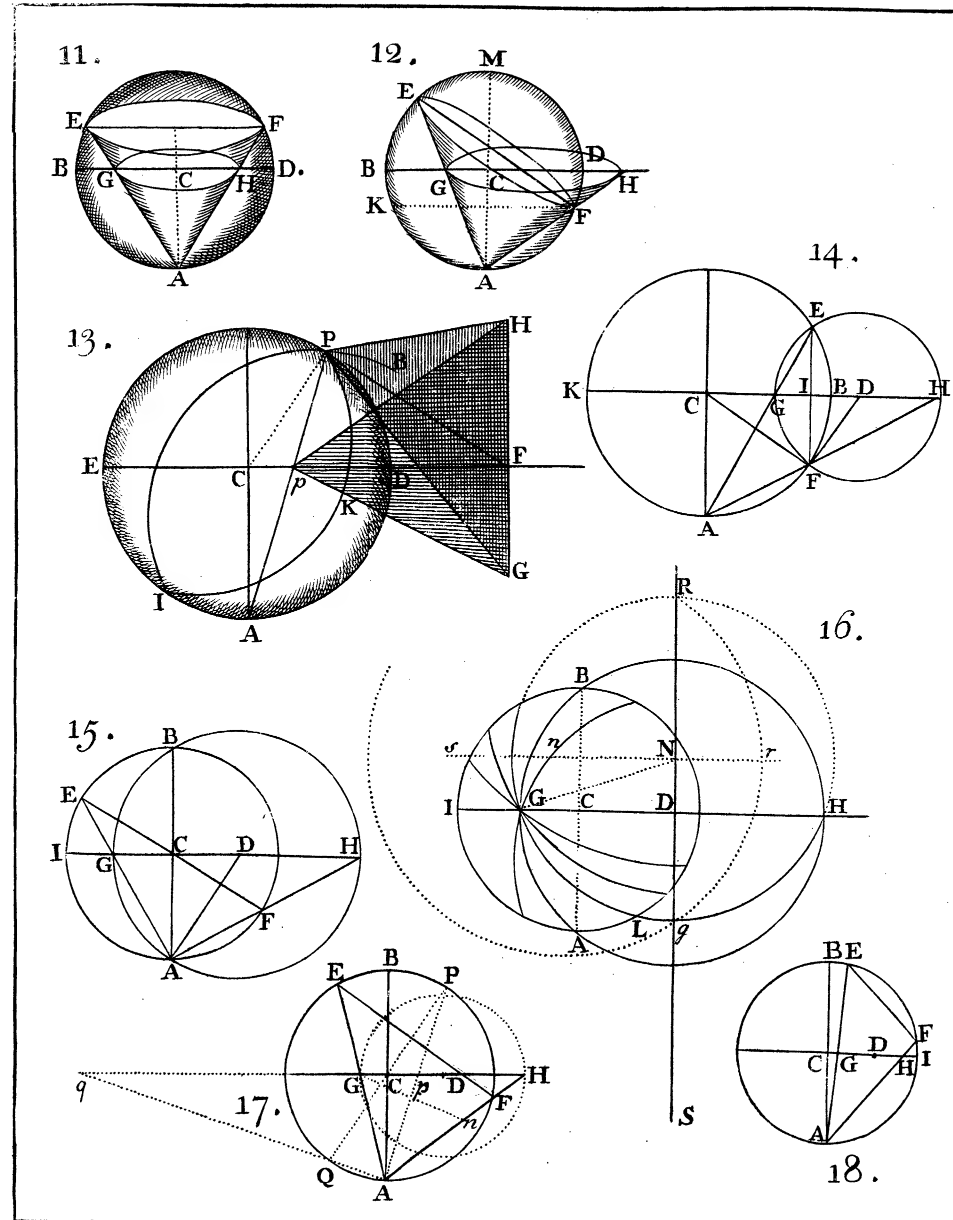
For since all great circles cut one another at a semicircle's distance, all circles passing thro' G must cut at the opposite point H; and therefore their centers must be in the Line RDS.

Cor. 2. *Hence also if any oblique circle* GLH *be required to make any given angle with another circle* BGAH, *it will be projected the same way with regard to* GAH *considered as a primitive, and* RS *its line of measures; as the circle* BGA *is on the primitive* BIA, *and line of measures* ID. *And therefore the tangent of the angle* AGL *to the radius* GD, *set from* D *to* N, *gives the center of* GL.

For the < NGD will then be equal to AGL, by Cor. 2. Prop. IV. and therefore GLH is rightly projected.

Cor. 3. *And for the same reason, if* N *be the center of the circle* GgHR; *the centers of all circles passing thro'* g *and* R, *will be in the line* rNs *perpendicular to* RS; *so* n *is the center of* grR. *But then as* g, R *do not represent opposite points of the circle* GgH, *therefore all circles passing thro'* g, R, (*as* grR) *will be lesser circles, except* GgHR.

SCHOLIUM.



Projection.

SCHOLIUM.

Fig. 16.

Of all great circles in the projection, the primitive is the least. For the radius of any oblique great circle (being the secant of the inclination) is greater than the radius of the primitive; as the secant is always greater than the radius. Therefore every oblique great circle in the projection is greater than the primitive.

PROP. VII.

The projected extremities of the diameter of any circle, are in the line of measures, distant from the center of the primitive circle, the semi-tangents of its nearest and greatest distances from the pole of projection opposite to the projecting point.

For the diameter of the circle EF is projected 15.
into GH, from the projecting point A. But GC 17.
is the semi-tangent of EB, and CH the semi-tangent of BF. *Q. E. D.*

Cor. 1. *The points where an inclined great circle* 15.
cuts the line of measures, within and without the primitive, is distant from the center of the primitive, the tangent and co-tangent of half the complement of the circle's inclination to the primitive

For CG = tangent of half EB, or of half the complement of IE the inclination. And (because the ∠ EAF is right) CH is the co-tangent of GAC or half EB.

Cor. 2. *Hence the center* D *of a projected circle is* 17.
in the line of measures distant, from the center of the 18.
primitive, half the difference of the semi-tangents of its nearest and greatest distance from the opposite pole, if it encompasses that pole; but half the sum of the semi-tangents if it lye on one side the pole of projection.

 Cor.

Fig. Cor. 3. *And the radius is half the sum of the semi-tangents, if the circle encompasses the pole; or half the difference if it lyes on one side.*

17. Cor. 4. *Hence also if pq be the projected poles, it will be* qG : pG : : qH : pH.

For draw Gn parallel to qA, and since P, Q are the poles, therefore qAp is a right angle, and since the angles GAp and pAH are equal, and Gn perpendicular to Ap, therefore GA = An; whence by similar triangles qG : qH : : An or AG : AH : : Gp, pH, (Geom. II. 25.) And consequently the line qH is cut horomonically in the points G, p.

P R O P. VIII.

The projected poles of any circle are in the line of measures, within and without the primitive, and distant from its center the tangent and co-tangent of half its inclination to the primitive.

19. The poles P, p of the circle EF are projected into D and d; and CD is the tangent of CAD or half BCP, that is, of half GCI, the inclination of the circle ICK, parallel to EF. Likewise Cd is the tangent of CAd, or the co-tangent of CAD.

Q. E. D.

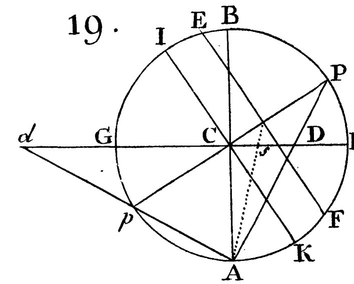
Cor. 1. *The pole of the primitive is its center; and the pole of a right circle is in the primitive.*

19. Cor. 2. *The projected center of any circle is always between the projected pole (nearest to it on the sphere) and the center of the primitive; and the projected centers of all circles lye between the projected poles.*

For the middle point of EF or its center is projected into S; and all the points in Pp (in which are all the centers) are projected into Dd.

Cor.

19.



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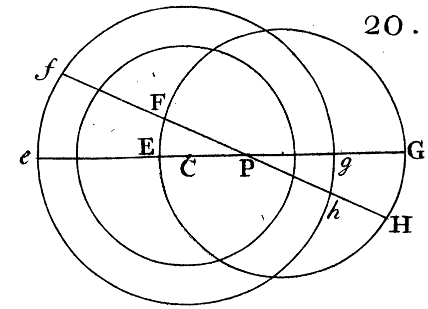

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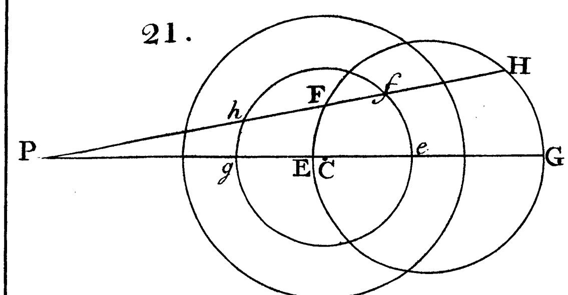

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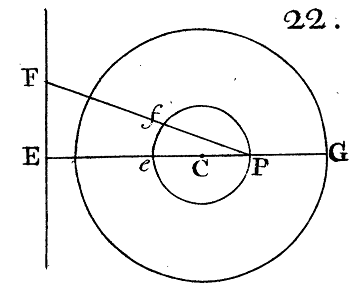

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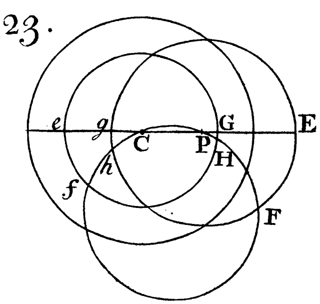

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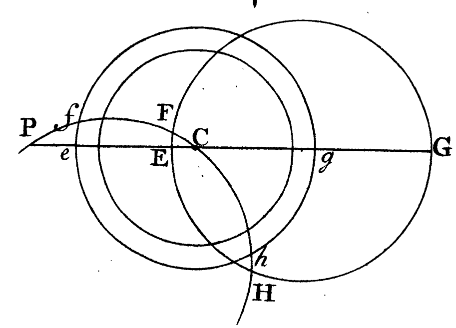


Cor. 3. *If* P *be the projected center of any circle* Fig.
EFG, *any right lines* EG, FH *passing thro'* P *will* 20.
intercept equal arches EF, GH.

For in any circle of the sphere, any two lines, passing thro' the center, intercept equal arches; and these are projected into right lines, passing thro' the projected center P, and therefore EF, GH, represent equal arches.

PROP. IX.

If EFGH, *efgh represent two equal circles, where-* 20.
of EFG *is as far distant from its pole* P, *as efg is* 21.
*from the projecting point. I say, any two right lines (e*EP, *and f*FP,) *being drawn thro'* P, *will intercept equal arches (in representation) of these circles; on the same side, if* P *falls within the circles; but on the contrary side, if without; that is,* EF = *ef, and* GH = *gh.*

For by the nature of the section of a sphere; any two circles passing thro' two given points or poles on the surface of the sphere, will intercept equal arches of two other circles equidistant from these poles. Therefore the circles EFG and *efg* on the sphere, are equally cut by the planes of any two circles passing thro' the projecting point and the pole P, on the sphere. But these circles (by Prop. I.) are projected into the right lines P*e* and P*f*, passing thro' P. And the intercepted arches, representing equal arches on the sphere, are therefore equal, that is, EF = *ef*, and GH = *gh*.

Cor. 1. *If a circle is projected into a right line* EF, 22.
perpendicular to the line of measures EG; *and if from the center* C *a circle ef*P *be described passing thro' its pole* P, *and* P*f be drawn; then arch ef* = EF. *And if any other circle be described whose vertex is* P, *the arch ef will always be equal to* EF.

Fig. Cor. 2. *Hence also, if from the pole of a great circle there be drawn two right lines, the intercepted arch of the projected great circle will be equal to the intercepted arch of the primitive.*

23. Cor. 3. *After the same manner, if there be two*
24. *equal circles* EF, *ef, whereof one is as far from the pole* P, *as the other is from the pole of projection e, opposite to the projecting point. Then any circle drawn thro' the points* P, C, *will intercept equal arch* EF $=$ *ef*; *and* GH $=$ *gh, between it and the line of measures* PCG.

For this is true on the sphere, and their projections are the same.

Cor. 4. *If from an angular point be drawn two right lines thro' the poles of its sides; the intercepted arch of the primitive, will be equal to that angle.*

For the distance of the poles is equal to that angle.

PROP. X.

25. *If* QH, NK *be two equal circles, whereof* NK
26. *is as far from the projecting point as* QH *from its pole* P; *and if they be projected into the circles whose radii are* MC *or* CL, *and* DF *or* FG, F *being the center of* DG, *and* E *the projected pole. I say, the pole* E *will be distant from their centers in proportion to the radii of the circles; that is,* CE : EF : : CL : DF *or* FG.

For since NK and ML are parallel, and arch NI $=$ PH, therefore $<$ ELI $=$ NKI (or *n*KI) $=$ GIP; therefore the triangles IEL and IEG are similar, whence EL : EI : : EI : EG. Again the angle EMI $=$ KNI $=$ PIQ, and therefore the triangles IEM and IED are similar, whence EM : EI : : EI : ED. Therefore $EI^2 = EL \times EG =$ EM

EM × ED. Consequently EM : EL : : EG : ED ; Fig
and by composition $\frac{EM + EL}{2} : \frac{EM - EL}{2} : :$
$\frac{EG + ED}{2} : \frac{EG - ED}{2}$; that is, CM : EC : : FG :
FF. *Q. E. D.*

Cor. 1. *Hence if the circle* KN *be as far from* 25.
the projecting point, as QH *is from either of its poles,* 26.
and if E, O, *be its projected poles; then will* EL :
EM : : ED : : EG : : OD : OG.

This follows from the foregoing demonstration, and Cor. 4. Prop. VII.

Cor. 2. *Hence also if* F *be the center, and* FD *the* 25.
radius of any circle QH, *and* E, O *the projected* 26.
poles; then EF : DF : : DF : FO.

For it follows from Cor. 1, that $\frac{EG + ED}{2} : \frac{EG - ED}{2} : : OG + OD : \frac{OG - OD}{2}$.

Cor. 3. *Hence if the circle* DBG, *be as far from* 27.
its projected pole P, as LMN *is from the projecting* 28.
point; and if any right lines be drawn thro' P, *as* MPG, NPK, *they will cut off similar arches* GK, MN *in the two circles.*

For from the centers C, F, draw the lines CN, FK, then since the angles CPN, and FPK are equal, and by this Prop. CP : CN : : FP : FK; therefore (Geom. II. 16.); the triangles PCN and PFK are similar; and the angle PCN = < PFK; therefore the arches MN and GK are similar.

Cor. 4. *Hence also if thro' the projected pole* P *of* 27.
any circle DBG, *a right line* BPK *be drawn. Then* 28.
I say the degrees in the arch GK *shall be the measure of* DB *in the projection. And the degrees in* DB, *shall be the measure of* GK *in the projection.*

For

Fig. For (by Prop. IX.) the arch MN is the meaſure of DB, and therefore GK which is ſimilar to MN, will alſo be the meaſure of it.

Cor. 5. *The centers of all projected circles are all beyond the projected poles (in reſpect to the center of the primitives); and none of their centers can fall between them.*

20. Cor. 6. *Hence it follows (by* Cor. 5. *and* Pr. VIII. Cor. 3.*) that all circles that are not parallel to the primitive have equal arches on the ſphere repreſented by unequal arches on the plane of projection.*

For if P be the projected center, then GH is greater than EF.

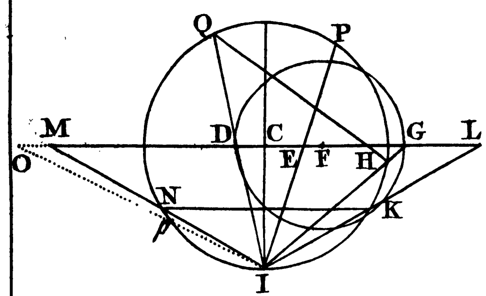
SCHOLIUM.

It will be eaſy by the foregoing propoſitions to deſcribe the repreſentation of any circle, and the reverſe will eaſily ſhow what circle of the ſphere any projected circle repreſents. What follows hereafter is deduced from the foregoing propoſitions, and will eaſily be underſtood without any other demonſtration.

If the ſphere was to be projected on any plane parallel to the primitive, 'tis all the ſame thing. For the cones of rays iſſuing from the projecting point, are all cut by parallel planes into ſimilar ſections, it only makes the projections bigger or leſs, according to the diſtance of the plane of projection, whilſt they are ſtill ſimilar; and amounts to no more than projecting from different ſcales upon the ſame plane. And therefore the projecting the ſphere on the plane of a leſſer circle is only projecting it upon the great circle parallel thereto, and continuing all the lines of the ſcheme to that leſſer circle.

PROP.

25.



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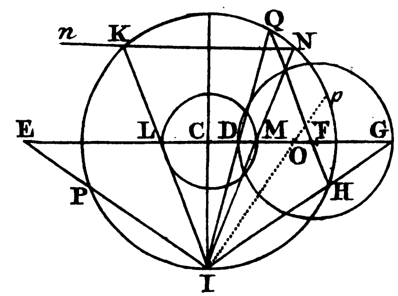

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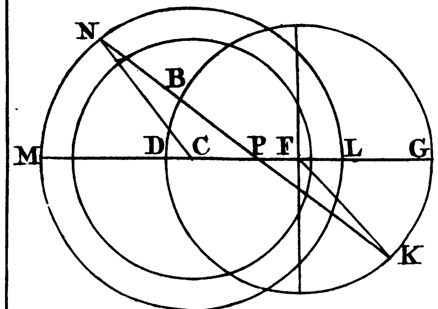

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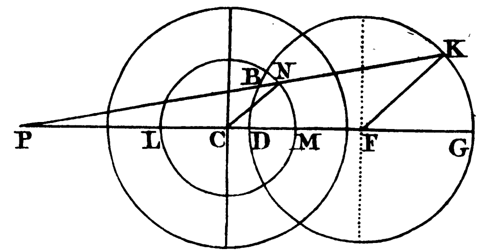

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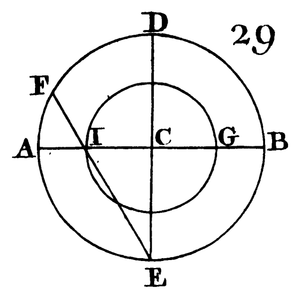

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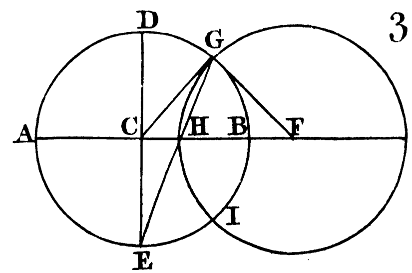


Projection.

IV. *p.* 22.

PROP. XI. *Prob.*

To draw a circle parallel to the primitive at a given distance from its pole.

Rule.

Thro' the center C draw two diameters AB, DE, perpendicular to one another. Take in your compasses the distance of the circle from the pole of the primitive opposite to the projecting point, and set it from D to F; from E draw EF to intersect AB in I; with the radius CI, and center C, describe the circle GI required. 29.

By the plain Scale.

With the radius CI, equal to the semi-tangent of the circles distance from the pole of projection opposite the projecting point, describe the circle IG. Here the radius of projection CA, is the tangent of 45°, or the semi-tangent of 90°.

PROP. XII. *Prob.*

To draw a lesser circle perpendicular to the primitive at a given distance from the pole of that circle.

Rule.

Thro' the pole B draw the line of measures AB, make BG the circle's distance from its pole, and draw CG, and GF perpendicular to it; with the radius FG describe the circle GI required. 30.

By the Scale.

Set the secant of the circle's distance from its pole from C to F, gives the center. With the tangent of that distance for a radius, describe the circle GI.

Or thus, make BG the circle's distance from its pole; and GF its tangent, set from G, gives F the center;

Fig. center; thro' G describe the circle GI from the cen-
30. ter F.

Cor. *Hence a great circle perpendicular to the primitive, is a right line* CDE *drawn thro' the center perpendicular to the line of measures.*

SCHOLIUM.

When the center F lyes at too a great a distance; draw EG, to cut AB in H; or lay the semi-tangent of DG from C to H. And thro' the three points G, H, I, draw a circle with a bow.

PROP. XIII. *Prob.*

To describe an oblique circle at a given distance from a pole given.

Rule.

31. Draw the line of measures AB thro' the given point *p*, if that point is given; and draw DE ⊥ to it, also draw E*p*P. Or if the point *p* is not given, set the height of the pole above the primitive from B to P. Then from P set of PH = PI = circle's distance from its pole; and draw EH, EI, to intersect AB in F and G. About the diameter FG describe the circle required.

By the Scale.

If the point P is given, apply C*p* to the semi-tangents and it gives the distance of the pole from D, the pole of projection opposite to the projecting point. This distance being had, you'll easily find the greatest and nearest distances of the circle from the pole of the primitive opposite to the projecting point; take the semi tangents of these distances and set from C to G and F, both the same way if the circle lye all on one side, but each its own way, if on different sides of D. And then FG is the diameter of the circle required to be drawn.

Cor.

Cor. 1. *If* F *be the pole of a great circle as of* Fig.
DLE. *Draw* EFH, *and make* HP = DH, *and* 31.
draw E*p*P, *and then P is its center.*

Or thus, draw EFH *thro' the pole* F, *make* HK 90 *degrees; draw* EK *cutting the line of meaſures in* L. *Thro' the three points* D, L, E, *draw the great circle required.*

Cor. 2. *Hence it will be eaſy to draw one circle parallel to another.*

PROP. XIV. *Prob.*

Thro' two given points A, B, *to draw a great circle.*

Rule.

Thro' one of the points A, draw a line thro' the 32.
center, ACG; and EF perpendicular to it. Then draw AF, and EG perpendicular to it. Thro' the three points A, B, G draw the circle required.

Or thus; From E (found as before) draw EH, and then HCI, and laſtly EIG, gives G a third point, thro' which the circle muſt paſs.

By the Scale.

Draw ACG; and apply AC to the ſemi-tangents, find the degrees, ſet the ſemi-tangent of its ſupplement from C to G, for a third point.

Or thus; Apply AC to the tangents, and ſet the tangent of its complement from C to G. And thro' the three points ABG, deſcribe the circle required.

For ſince HEI or AEG is a right angle, therefore A, G are oppoſite points of the ſphere; and therefore all circles paſſing thro' A and G are great circles.

SCHOLIUM.

If the points A, B, G lie nearly in a right line, then you may draw a circle thro' them with a bow.

PROP.

Fig.

PROP. XV. *Prob.*

About a pole given, to describe a circle thro' a given point.

Rule.

33. Let P be the pole, and B the given point; thro' P, B describe the great circle AD (by Prop. XIV.), whose center is E; thro' the center C draw CPH; and from the center E, draw EB, and BF perpendicular to it. To the center F, and radius FB describe the circle BGH required.

PROP. XVI. *Prob.*

To find the poles of any circle FNG.

Rule.

31. Thro' its center draw the line of measures AG, and DE perpendicular to it. Draw EFH, and set its distance (from its own pole) from H to P, and draw EpP, then p is the pole.

Or thus, Draw EFH, EIG, and bisect HI in P, and draw EpP, and p is the internal pole. Lastly draw PCQ, and EQq, and q is the external pole.

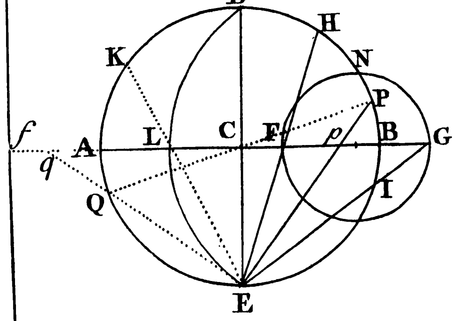
In a great circle DLE, draw ELK, and make DH = AK, (or KH = AD, and draw EFH, and F is the pole.

By the Scale.

Apply CF to the semi-tangents, and note the degrees. Take the sum of these degrees and of the circle's distance from its pole, if the circle lie all on one side, but their difference if it encompasses the pole of projection; set the semi-tangent of this sum or difference from C to the internal pole p. And the semi-tangent of its supplement Cq, gives the external pole q.

Or thus, Apply CF and CG to the semi-tangents, set the semi-tangent of half the sum of the degrees (if

31.



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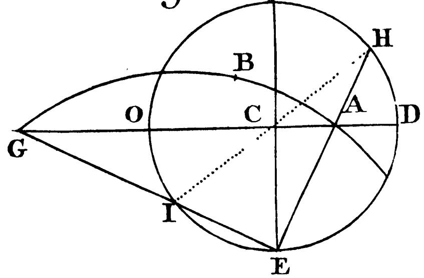

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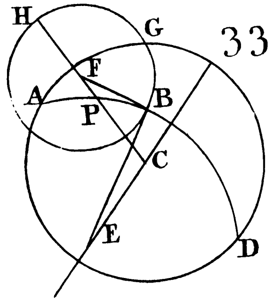

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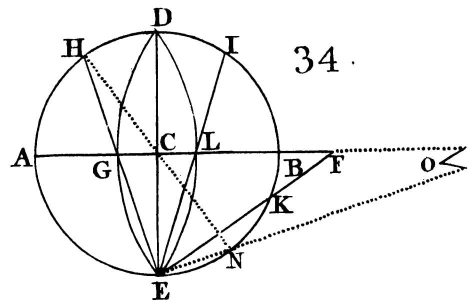

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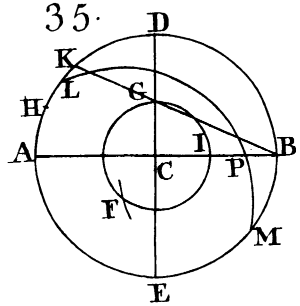

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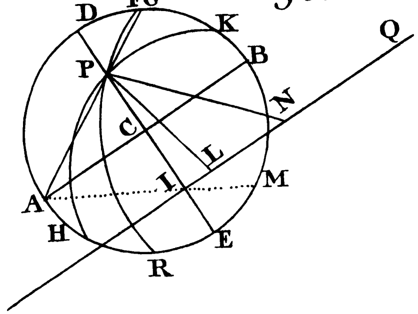


Projection.

V. p. 26.

(if the circle lies all one way) or of half the difference (if it encompasses the pole of projection), from C to the pole p; and the semi-tangent of the supplement, Cq gives the external pole q. Fig. 31.

In a great circle as DLE, draw the line of measures AB perp. to DE; and set the tangent and co-tangent of half its inclination, from the center C, different ways to F and f; which gives the internal and external poles F and f.

PROP. XVII. *Prob.*

To draw a great circle at any given inclination above the primitive; or making any given angle with it, at a given point.

Rule.

Draw the line of measures AB; and DCE perpendicular to it. Make EK = 2HD = twice the complement of the circle's inclination; (or DK = 2AH = twice the inclination); and draw EKF, then F is the center of EGD, the circle required. 34.

Or thus; Draw DE and AB perp. to it, and let D be the point given. Make AH the inclination, and draw EGH and HCN; and ENO, to cut AB in O. Then bisect GO in F, for the center of the circle required.

By the Scale.

Set the tangent of the inclination in the line of measures from C to F, then F is the center. Set the semi-tangent of the complement from C to G; then GF or DF is the radius.

Or the secant of the inclination set from G or D to F gives the center.

Cor. *To draw an oblique circle to make a given angle with a given oblique circle* DGE *at* D. *Draw* EGH, *and set the given angle from* H *to* I, *and draw* ELI. *Thro'* D, L, E *describe a great circle.*

PROP.

PROP. XVIII. *Prob.*

Through a given point P, *to draw a great circle; to make a given angle with the primitive.*

Rule.

35. Thro' the point given P and the center C draw the line AB; and DE perpendicular to it. Set the given angle from A to H and from H to K, and draw BGK; with radius CG, and center C deſcribe the circle GIF; and with radius BG and center P croſs that circle in F. Then with radius FP and center F, deſcribe the circle LPM required.

By the Scale.

With the tangent of the given angle and one foot in C, deſcribe the arch FG. With the ſecant of the given angle and one foot in the given point P, croſs that arch at F. From the center F deſcribe a circle thro' the point P.

PROP. XIX. *Prob.*

To draw a great circle to make a given angle with a given oblique circle FPR, *at a given point* P, *in that circle.*

Rule.

36. Thro' the center C and the given point P, draw the right line DE; and AB perpendicular to it; draw APG and make BM = 2DG; and draw AM to cut DE in I. Draw IQ perpendicular to DE, then IQ is the line wherein the centers of all circles are found which paſs thro' the point P. Find N the center of the given circle FPR, and make the angle NPL equal to the given angle, then L is the center of the circle HPK required.

By

By the Scale. Fig.

Thro' P and C draw DE; apply CP to the semi-tangents, and set the tangent of its complement from C to I (or the secant from P to I). On DI erect the perpendicular IQ. Find the center N of FPR, and make the angle NPL = angle given, and L is the center. 36.

Cor. *If one circle is to be drawn perpendicular to another, it must be drawn thro' its poles.*

PROP. XX. *Prob.*

To draw a great circle thro' a given point P, *to make a given angle with a given great circle* DE.

Rule.

About the given point P as a pole (by Prop. 13. Cor. 1.) describe the great circle FG; find I the pole of the given circle DE, and (by Prop. 16.) about the pole I (by Prop. 13.) describe the small circle HKL at a distance equal to the given angle, to intersect FG in H; about the pole H describe (by Prop. 13.) the great circle APB required. 37.

PROP. XXI. *Prob.*

To draw a great circle to cut two given great circles abd, ebf at given angles.

Rule.

Find the poles *s*, *r*, of the two given circles, by Prop. 16. about which draw two parallels *phk*, *pnk*, at the distances respectively equal to the angles given by Prop. 13. the point of intersection P, is the pole of the circle *moq* required. 50.

Cor. *Hence, to draw a right circle to make with an oblique circle, abd, any given angle. Draw a parallel phk at a distance from the pole of the oblique circle, equal to the given angle. Its intersection*

Fig. *with the primitive, gives the pole of the right circle get required.*

P R O P. XXII. *Prob.*

To lay any number of degrees on a great circle, or to measure any arch of it.

Rule.

38. Let AFI be the primitive; find the internal pole P of the given circle DEH (by Prop 16.) lay the degrees on the primitive from A to F, and draw PA, PF, intercepting the part required DE. Or to meaſure DE, draw PEF and PDA, and AF is its meaſure, and applied to the line of chords ſhows how many degrees it is.

Or thus; Find the external pole *p* of the given circle, ſet the given degrees from I to K, and draw *p*I, *p*K, intercepting the part DE required. Or to meaſure DE, thro' D and E draw *p*I, *p*K, then KI is the meaſure of DE.

Or thus; Thro' the internal pole P, draw the lines DPG, and EPL; ſetting the given degrees from G to L in the circle GL; then DE is the arch required. Or if DE be to be meaſured, then the degrees in the arch GL is the meaſure of DE.

Or thus; Set the given degrees from G to H in the circle GL and from the external pole P, draw *p*G, *p*H, intercepting DE the arch required Or to meaſure DE, draw *p*DG, *p*EH, then the degrees in GH, is equal to DE.

By the Scale for right Circles.

38. Let CA be the right circle, take the number of degrees off the ſemi-tangents and ſet from C to D for the arch CD. Or if the given degrees are to be ſet from A, then take the degrees off the ſemi-tangents from 90° towards the beginning, and ſet from A to D. And if CD was to be meaſured, apply

apply it to the beginning of the semi-tangents; and Fig. to measure AD, apply it from 90° backwards, and the degrees intercepted gives its measure.

SCHOLIUM.

The primitive is measured by the line of chords, or else it is actually divided into degrees.

PROP. XXIII. *Prob.*

To set any number of degrees on a lesser circle, or to measure any arch of it.

Rule.

Let the lesser circle be DEH; find its internal pole 38. P, by Prop. 16. describe the circle AFK parallel to the primitive, by Prop. 11. and as far from the projecting point, as the given circle DE is from its internal pole P, set the given degrees from A to F, and draw PA, PF intersecting the given circle in D, E; then DE is the arch required. Or to measure DE, draw PDA, PEF, and AF shows the degrees in DE.

Or thus; Find the external pole *p*, of the given circle by Prop. 16. describe the lesser circle AFK as far from the projecting point, as DE the given circle is from its pole *p*, by Prop. 11. set the degrees from I to K and draw *p*DI, *p*EK, then DE represents the given number of degrees. Or to measure DE; draw *p*DI, *p*EK; and KI is the measure of DE.

Or thus; Let O be the center of the given circle DEH; thro' the internal pole P, draw lines DPG, EPL, divide the quadrant GQ into 90 equal degrees, and if the given degrees be set from G to L, then DE will represent these degrees. Or the degrees in GL will measure DE.

Or thus; Divide the quadrant GR into 90 equal parts or degrees, and set the given degrees from G to H, and draw *p*DG, *p*EH, from the external pole *p*; then DE will represent the given degrees. Or

Fig. thro' D, E drawing pDG, pEH, then the number of equal degrees in GH is the meaſure of DE.

SCHOLIUM.

Any circle parallel to the primitive is divided or meaſured, by drawing lines from the center, to the like diviſions of the primitive. Or by help of the chords on the ſector, ſet to the radius of that circle.

PROP. XXIV. *Prob.*

To meaſure any angle.

Rule.

By Cor. 1. Prop. 13. About the angular point as a pole, deſcribe a great circle, and note where it interſects the legs of the angle; thro' theſe points of interſection, and the angular point, draw two right lines, to cut the primitive; the arch of the primitive intercepted between them is the meaſure of the angle. This needs no example.

Or thus; by Prop. 16. Find the two poles of the containing ſides, (the neareſt, if it be an acute angle, otherwiſe the furtheſt) and thro' the angular point and theſe poles, draw right lines to the primitive, then the intercepted arch of the primitive is
31. the angle required. As if the angle AEL was required. Let C and F be the poles of EA and EL. From the angular point E, draw ECD and EFH. Then the arch of the primitive DH, is the meaſure of the angle AEL.

SCHOLIUM.

Becauſe in the Stereographic Projection of the Sphere, all circles are projected either into circles or right lines, which are eaſily deſcribed; therefore this ſort of projection is preferred before all others. Alſo thoſe planes are preferred before others to project upon, where moſt circles are projected into right lines, they being eaſier to deſcribe and meaſure than circles are; ſuch are the projections on the planes of the meridian and ſolſtitial colure.

SECT.

Fig.

SECT. III.

The Gnomonical Projection of the SPHERE.

PROP. I.

Every great circle as BAD *is projected into a right line, perpendicular to the line of measures, and distant from the center, the co-tangent of its inclination, or the tangent of its nearest distance from the pole of projection.* 39.

Let CBED be perpendicular both to the given circle BAD and plane of projection, and then the intersection CF will be the line of measures. Now since the plane of the circle BD, and the plane of projection are both perpendicular to BCDE, therefore their common section will also be perpendicular to BCDE, and consequently to the line of measures CF. Now since the projecting point A is in the plane of the circle, all the points of it will be projected into that section; that is, into a right line passing thro' *d*, and perpendicular to C*d*. And C*d* is the tangent of CD, or co-tangent of C*d*A. *Q. E. D.*

Cor. 1. *A great circle perpendicular to the plane of projection is projected into a right line passing thro' the center of projection; and any arch is projected into its correspondent tangent.* 39.

Thus the arch CD is projected into the tangent C*d*.

Cor. 2. *Any point as* D, *or the pole of any circle, is projected into a point d distant from the pole of projection* C, *the tangent of that distance.*

Cor. 3. *If two great circles be perpendicular to each other, and one of them passes thro' the pole of projec-*

 tion;

Fig. *tion; they will be projected into two right lines per-*
39. *pendicular to each other.*

For the representation of that circle which passes thro' the pole of projection is the line of measures of the other circle.

Cor. 4. *And hence if a great circle be perpendicular to several other great circles, and its representation pass thro' the center of projection; then all these circles will be represented by lines parallel to one another, and perpendicular to the line of measures or representation of that first circle.*

PROP. II.

39. *If two great circles intersect in the pole of projection; their representations shall make an angle at the center of the plane of projection equal to the angle made by these circles on the sphere.*

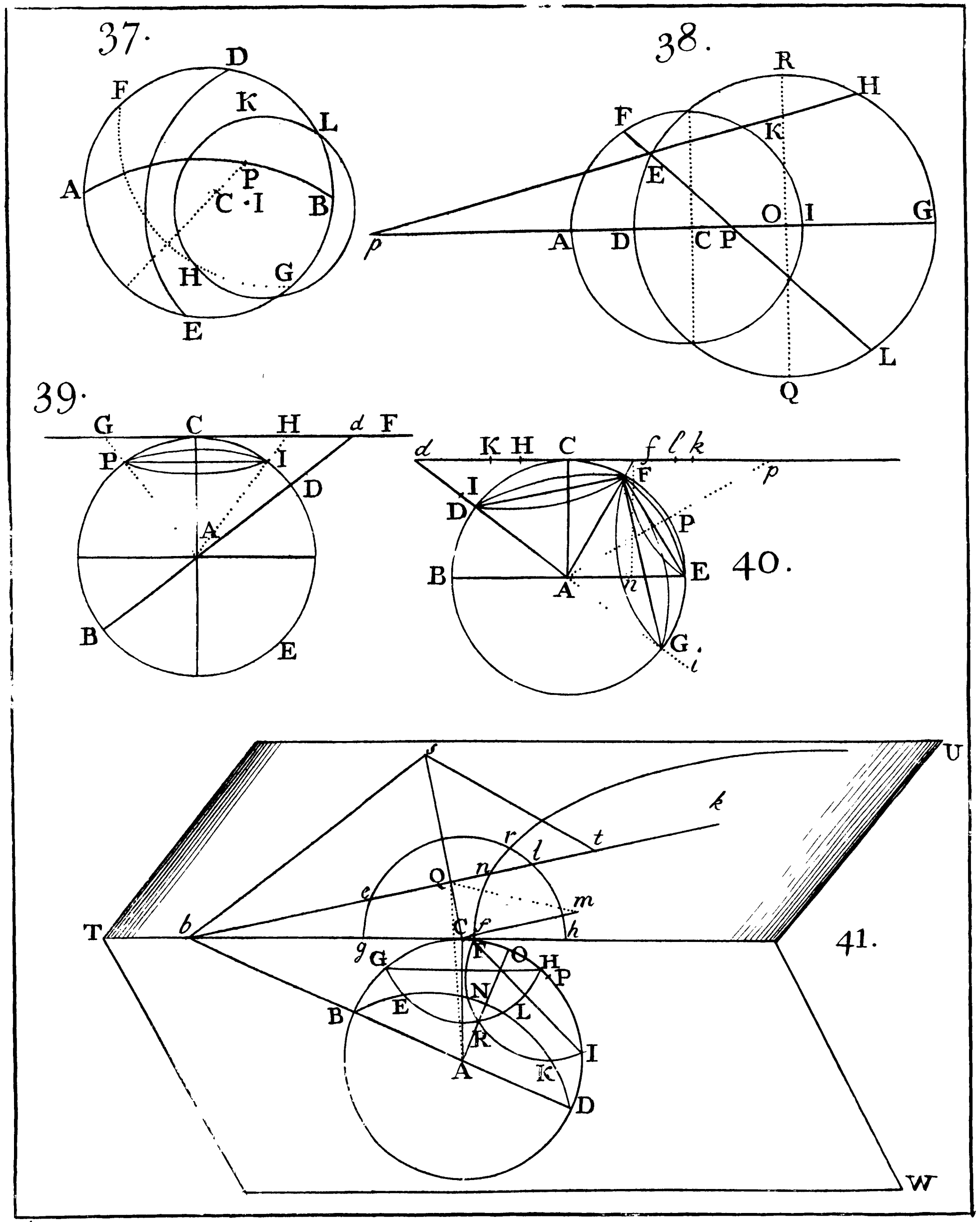
For since both these circles are perpendicular to the plane of projection; the angle made by their intersections with this plane, is the same as the angle made by these circles. *Q. E. D.*

PROP. III.

Any lesser circle parallel to the plane of projection is projected into a circle, whose center is the pole of projection; and radius the tangent of the circle's distance from the pole of projection.

39. Let the circle PI be parallel to the plane GF, then the equal arches PC, CI are projected into the equal tangents GC, CH; and therefore C the point of contact and pole of the circle PI and of the projection, is the center of the representation GH. *Q. E. D.*

Cor. *If a circle be parallel to the plane of projection, and 45 degrees from the pole, it is projected into a circle*



Projection. VI. p. 34.

a circle equal to a great circle of the sphere; and may Fig.
therefore be looked upon as the primitive circle in this projection, and its radius the radius of projection.

PROP. IV.

Every lesser circle (not parallel to the plane of projection) is projected into a conic section, whose transverse axis is in the line of measures, and whose nearest vertex is distant from the center of the plane the tangent of its nearest distance from the pole of projection; and the other vertex is distant the tangent of its furthest distance. 40.

Let BE be parallel to the line of measures *dp*, then any circle is the base of a cone whose vertex is at A, and therefore that cone being produced will be cut by the plane of projection in some conic section; thus the circle whose diameter is DF will be cut by the plane in an ellipsis whose transverse is *df*; and C*d* is the tangent of CAD, and C*f* of CF. In like manner the cone AFE being cut by the plane, *f* will be the nearest vertex; and the other point into which E is projected is at an infinite distance. Also the cone AFG (whose base is the circle FG) being cut by the plane *f* is the nearest vertex; and GA being produced gives *d* the other vertex. *Q. E. D.*

Cor. 1. *If the distance of the furthest point of the circle be less than 90° from the pole of projection, then it will be projected into an ellipsis.*

Thus DF is projected into *df*, and DC being less than 90°, the section *df* is an ellipsis, whose vertices are at *d* and *f*; for the plane *df* cuts both sides of the cone, *d*A, *f*A.

Cor. 2. *If the furthest point be more than 90 degrees from the pole of projection, it will be projected*

 into

Fig. 40. *into an hyperbola. Thus the circle* FG *is projected into an hyperbola whose vertices are* f *and* d, *and transverse* fd.

For the plane dp cuts only the side Af of the cone.

Cor. 3. *And in the circle* EF, *where the furthest point* E *is* 90° *from* C; *it will be projected into a parabola, whose vertex is* f.

For the plane dp (cutting the cone FAE) is parallel to the side AE.

Cor. 4. *If* H *be the center, and* K, k, l, *the focus of the ellipsis, hyperbola, or parabola; then* $HK = \frac{Ad - Af}{2}$ *for the ellipsis, and* $Hk = \frac{Ad + Af}{2}$ *for the hyperbola; and (drawing* fn *perpendicular on* AE*)* $fl = \frac{nE + Ff}{2}$, *for the parabola; which are the representations of the circles* DF, FG, FE *respectively.*

This all appears from the Conic Sections.

PROP. V.

41. *Let the plane* TW *be perpendicular to the plane of projection* TV, *and* BCD *a great circle of the sphere in the plane* TW. *And let the great circle* BED *be projected in the right line* bek. *Draw* CQS $\perp$ bk, *and* Cm $\parallel$ *to it and equal to* CA, *and make* $QS = Qm$; *then I say any angle* $Qst = Qt$.

Suppose the hypothenuse AQ to be drawn, then since the plane ACQ is perpendicular to the plane Tv, and bQ is $\perp$ to the intersection CQ, therefore bQ is perpendicular to the plane ACQ, and consequently bQ is perpendicular to the hypothenuse AQ. But $AQ = Qm = Qs$, and Qs is also perpendicular to bQ. Therefore all angles made at S cut the line bQ in the same points as the angles made at at A; but by the angles at A the circle BED is projected into the line bQ. Therefore the angles at s are the measures of the parts of the projected circle bQ; and s is the dividing center thereof. *Q. E. D.* Fig. 41.

Cor. 1. *Any great circle tQb is projected into a line of tangents to the radius* SQ.

For Qt is the tangent of the angle QSt to the radius QS or Qm.

Cor. 2. *If the circle bC pass thro' the center of projection; then* A *the projecting point is the dividing center thereof. And Cb is the tangent of its correspondent arch* CB, *to* CA *the radius of projection.*

PROP. VI.

Let the parallel circle GEH *be as far from the pole of projection* C *as the circle* FKI *is from its pole* P; *and let the distance of the poles* C, P *be bisected by the radius* AO, *and draw* bAD *perpendicular to* AO; *then any right line bek drawn thro' b, will cut off the arches* hl = Fn, *and* $ge = kf$ (*supposing f the other vertex*), *in the representations of these equal circles in the plane of projection.* 41.

For let G, E, R, L, H, N, R, K, I be respectively projected into the points g, e, r, l, h, n, r, k, f. Then since in the sphere, the arch BF = DH, and arch BG = DI. And the great circle BEKD makes the angles at B and D equal, and is projected into a right line as bl; therefore the triangular figures BFN and DHL are similar, and equal; and likewise BGE, and DIK are similar and equal, and LH = NF, and KI = EG; whence it is evident their projections lh = nF, and $kf = ge$. *Q. E. D.*

PROP.

Fig.

PROP. VII.

42. *If blg and Fnk be the projections of two equal circles, whereof one is as far from its pole P as the other from its pole C; which is the center of projection; and if the distance of the projected poles C, p be divided in o, so that the degrees in Co, op, be equal, and the perpendicular oS be erected to the line of measures gh. I say the lines pn, Cl, drawn from the poles C, p thro' any point Q in the line oS, will cut off the arch Fn = < QCp.*

For drawing the great circle GPI, in a plane perpendicular to the plane of projection. The great circle AO perpendicular to CP is projected into *o*S by Prop. I. Cor. 3. Now let Q be the projection of *q*, and since *p*Q, CQ are right lines, therefore they represent the great circles P*q*, C*q*. But the spherical triangle P*q*C is an isoceles-triangle, and therefore the angles at P and C are equal. But because P is the pole of FI, therefore the great circle P*q* continued, will cut an arch off FI = < CP*q* = < PC*q* = < QC*p* by Prop. II. That is (since F*n* represents the part cut off from FI) arch F*n* = arch *lh* or < QC*h*. *Q. E. D.*

Cor. *Hence if from the projected pole p of any circle, a perpendicular be erected to the line of measures; it will cut off a quadrant from the representation of that circle.*

For that perpendicular will be parallel to OS; Q being at an infinite distance.

PROP.

PROP. VIII.

Let Fnk be the projection of any circle FI, and p the projected pole P. And if Cg be the co-tangent of CAP, and gB perpendicular to the line of measures gC, and CAP be bisected by AO, and the line oB, be drawn to any point B, and also pB cutting Fnk in d. I say the angle goB = arch Fd. 42.

For the arch PG is a quadrant, and the $<goA = <gpA + <oAp =$ (because GCA and gAp are right angles) $gAC + oAp = gAC + CAo = <gAo$. Therefore $gA = go$, consequently o is the dividing center of gB the representation of GA; and consequently by Prop. V. $<goB$ is the measure of gB. But since pq represents a quadrant, therefore p is the pole of gB, and therefore the great circle pdB passing thro' the pole of the circles gB and Fn will cut off equal arches in both, that is $Fd = gB = <goB$. *Q. E. D.*

Cor. *The $<goB$ is the measure of the angle gpB.* For the triangle gpB represents a triangle on the sphere wherein the arch which gB represents is equal to the angle which $<p$ represents, because gp is 90 degrees. Therefore goB is the measure of both.

Scholium.

Thus far I have treated of the theory; what follows is the practical part, and depends altogether on what is above delivered, in which I think no difficulty can occur. In the Gnomonical Projection, the plane projected on, is supposed to touch the hemisphere to be projected, in its vertex; and the point of contact will be *the center of projection.* But if it be required to project upon any plane parallel

Fig. rallel to this touching plane, the proceſs will be no
42. way different, and is only taking a greater or leſſer radius of projection, according to the greater or leſſer diſtance; which is in effect projecting a greater or leſſer ſphere upon its touching plane.

When you have the ſphere to project this way, upon a given plane; it will aſſiſt the imagination, if you ſuppoſe yourſelf placed in the center of the ſphere with your face towards the plane, whoſe poſition is given; and from thence projecting with your eye, the circles of the ſphere upon this plane.

PROP. IX. *Prob.*

43. *To draw a great circle, thro' a given point, and at a given diſtance from the pole of projection.*

Rule.

Deſcribe the circle ADB with the radius of projection, and thro' the given point P draw the right line PCA, and CE perpendicular to it; make the angle CAE = given diſtance of the circle from C, and thro' E deſcribe the circle EFG, and thro' P draw the line PK touching the circle in I, then is PIK the circle required.

By the plain Scale.

With the tangent of the circle's diſtance from the pole of projection C, deſcribe the circle EIF, and draw PK to touch this circle; and PIK is the circle required.

PROP. X. *Prob.*

3. *To draw a great circle perpendicular to a given great circle, which paſſes thro' the pole of projection; and at a given diſtance from that pole.*

Rule.

Draw the primitive ADB. Let CI be the given circle, draw CL perpendicular to CI, and make the angle

angle CLI = the given diſtance; thro' I draw KP Fig.
parallel to CL for the circle required. 43.

By the Scale.

In the given circle CI, ſet the tangent of the given diſtance, from C to I; thro' I draw KP perpendicular to CI, then KP is the circle required.

PROP. XI. *Prob.*

To meaſure any part of a great circle; or to ſet any 44.
number of degrees thereon.

Rule.

Let EP be the great circle; thro' C draw ID perpendicular to EP, and CB parallel to it. Let EBD be a circle deſcribed with the radius of projection CB, make IA = IB; then A is the dividing center of EP, conſequently drawing AP, the $\angle$ IAP = meaſure of the given arch IP.

Or if the degrees be given, make the $\angle$ IAP = theſe given degrees, which cuts off IP, the arch correſpondent thereto.

By the Scale.

Draw ICD perpendicular to EP; apply CI to the tangents, and ſet the ſemi-tangent of its complement from C to A, gives the dividing center of EP, &c.

PROP. XII. *Prob.*

To draw a great circle to make a given angle with 51.
a given great circle, at a given point; or to meaſure an angle made by two great circles.

Rule.

Let P be the given point, and PB the given great circle. Draw thro' P, and C the center of projection, the line PCG, to which from C draw CA perpendicular,

Fig. dicular, and equal to the radius of projection. Draw
51. PA and AG perpendicular to it, at G erect BD perpendicular to GC, cutting PB in B; draw AO bisecting the angle CAP; then at the point O, make BOD = angle given, and from D draw the line DP, then BPD is the angle required.

Or if the degrees in the angle BPD be required, from the points B, D, draw the lines BO, DO; and the angle BOD is the meafure of BPD.

Cor. *If an angle be required to be made at the pole or center of projection, equal to a given angle; this is no more than drawing two lines from the center making the angle required. And if one great circle be to be drawn ⊥ to another great circle, it muft be drawn thro' its pole.*

P R O P. XIII. *Prob.*

43. *To project a leffer circle parallel to the primitive.*

Rule.

With the radius of projection AC, and center C, defcribe the primitive circle ADB, by Cor. Prop. III. and draw ACB, and GCE perpendicular to it.

Set the circle's diftance from its pole from B to H, and from H to D, and draw AFD. With radius CE defcribe the circle EFG required.

By the Scale.

With the radius CE equal to the tangent of the circle's diftance from its pole, defcribe the circle EFG, for the circle required.

P R O P. XIV. *Prob.*

48. *To draw a leffer circle perpendicular to the plane of projection.*

Rule.

Thro' the center of projection C, draw its parallel great circle TI. At C make the angle ICN and

and TCO = the given circle's distance from its pa- Fig.
rallel great circle TI; make CL equal radius of 48.
projection, and draw LM perpendicular to CL. Set LM from C to V, or CM from C to F. Then thro' the vertex V between the assymptotes CN, CO describe the hyperbola WVK. Or to the focus F, and semi-transverse CV, describe the hyperbola; for the circle required.

Otherwise by Points.

Thro' the center of projection C draw the line of measures CF, and TCI perpendicular to it, draw any number of right lines CV, DE, GH, IK &c. and PQ, RS, TW, &c. perpendicular to TI. And by Prop. XI. make CV, DE, GH, &c. each equal to the distance of the given circle from its parallel great circle; then all the points W, S, Q, V, E, H, K, &c. joined by a regular curve will be the representation of the circle required.

Or thus.

Make the angle *iak* = distance of the given circle from its parallel great circle. Then thro' the center of projection C, draw the great circle TCI parallel to the circle given, upon which erect the perpendicular CA = radius of projection. Also draw any number of right lines CV, DE, GH, IK, &c. perpendicular to TI. Then take each of the distances from A to C, D, G, I, &c. and set them from *a* to *c*, *g*, *d*, *i*, &c. and to *ai* draw the perpendiculars *cv*, *de*, *gh*, *ik*, &c. and make CV, DE, GH, IK, &c. respectively equal to *cv*, *de*, *gh*, *ik*, &c. which gives the points V, E, H, K, &c. after the same manner on the other side, find the points Q, S, W, &c. then thro' all these points W, S, Q, V, E, H, K, &c. draw a regular curve, which will be an hyperbola representing the circle given.

By

Fig.

By the Scale.

48\. Take the tangent of the circle's diſtance from its parallel great circle, and ſet it from C (the center of projection) to V, and the ſecant thereof from C to F. Then with the ſemi-tranſverſe CV, and focus F, deſcribe the hyperbola WVHK.

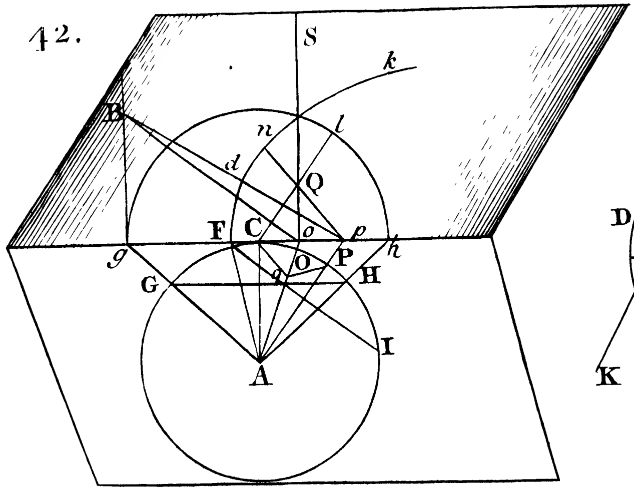
PROP. XV. *Prob.*

To project any leſſer oblique circle given.

Rule.

Draw the line of meaſures *dp*, and at C the
45\. center of projection draw CA ⊥ to *dp* and = radius of projection; with the center A, deſcribe the circle DCFG; and draw RAE parallel to *dp*. Then take the greateſt and leaſt diſtances of the circle from the pole of projection and ſet from C, to D and F, for the circle DF; and from A, the projecting point, draw AF*f*, and AD*d*, then *df* will be the tranſverſe axis of the ellipſis. But if D fall beyond the line RE, as at G, then draw a line from G backward thro' A to D, and then *df* is the tranſverſe of an hyperbola. But if the point D fall in the line RE as at E, then the line AE no where meets the line of meaſures, and the projection of E is at an infinite diſtance, and then the circle will be projected into a parabola whoſe vertex is *f*. Laſtly, biſect *df* in H the center, and for the ellipſis take half the difference of the lines A*d*, A*f*, and ſet from H to K for the focus. But for the hyperbola take half the ſum of A*d*, A*f*, and ſet from H to the focus *k* of the hyperbola. Then with the tranſverſe *df* and focus K or *k* deſcribe the ellipſis *d*M*f*, or the hyperbola *fm*. For the projection of the circle given.

But for the parabola make EQ = F*f*, and draw *fn* ⊥ AQ, and ſet ½*n*Q from *f* to K the focus. Then with



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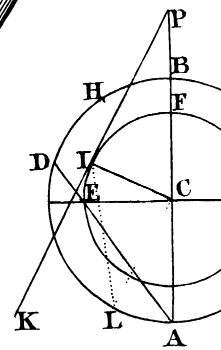

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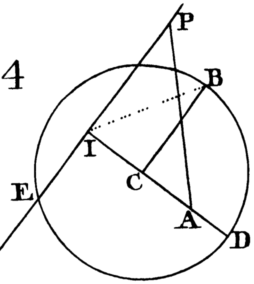

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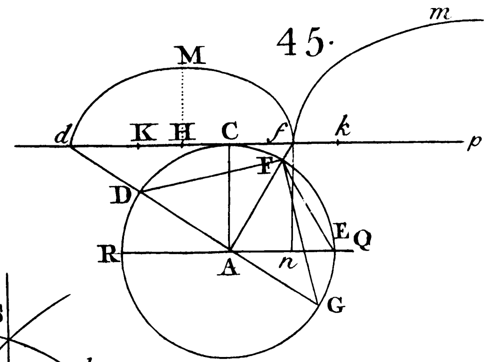

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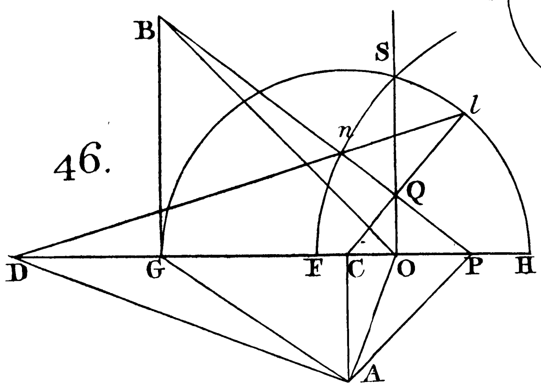

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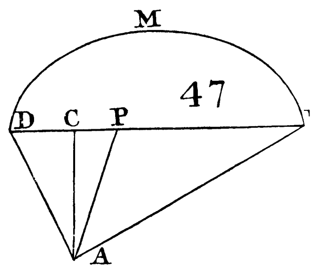
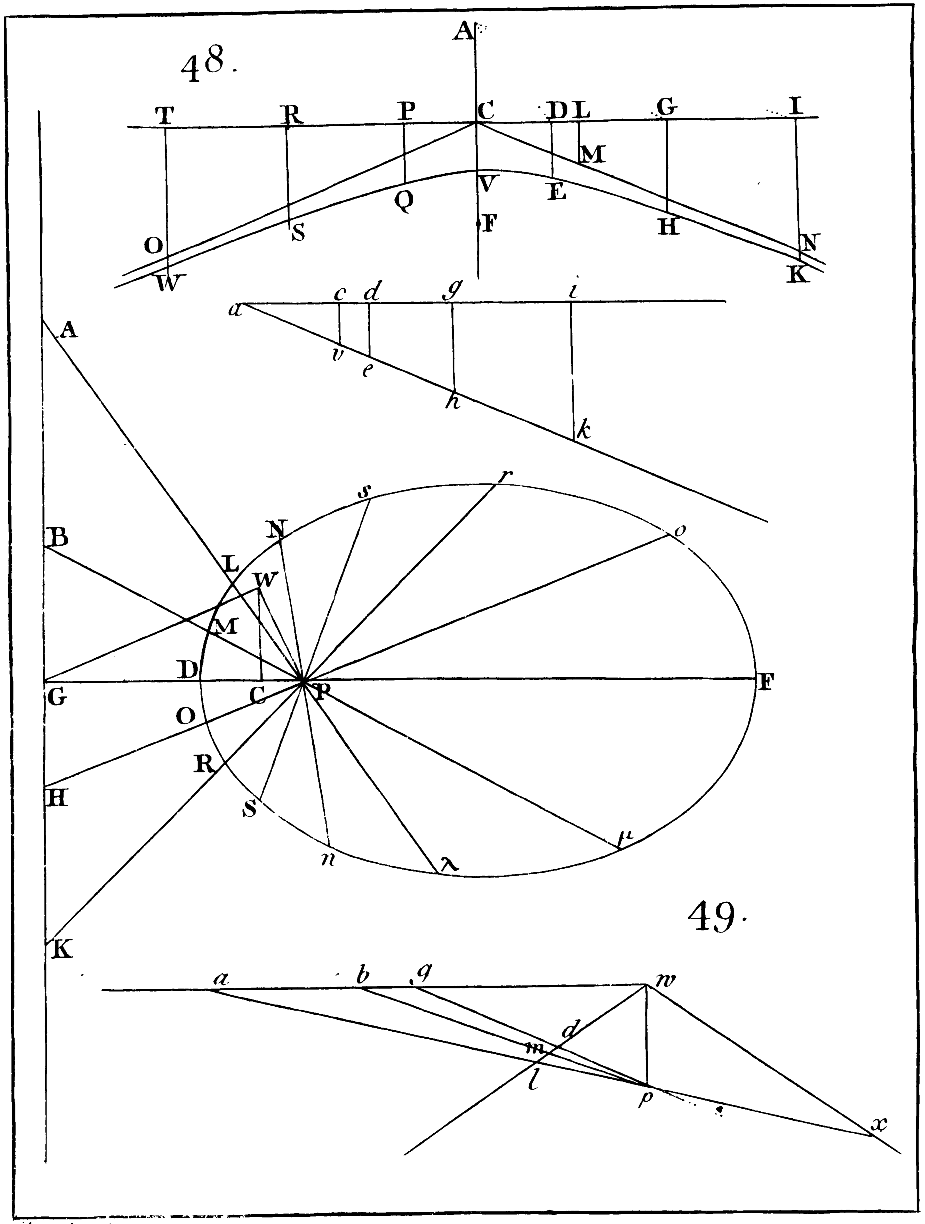


47

Projection.

VII pl.



Projection. VIII p. 44.

with the vertex f and focus k describe the parabola Fig.
fm, for the projection of the given circle FE.

Otherwise by Points.

Thro' the center of projection C, draw the line of measures CF, passing thro' the pole P (if P is given; but if not, find it, by setting off CP = the distance of that pole, from the center of projection, by Prop. XI.) then set off PD, PF equal to the given distance from its pole, by Prop. XI. Thro' P draw a sufficient number of right lines, Lλ, Mμ, Nn, Oo, Rr, Ss, &c. which will all represent great circles. Find the dividing centers of each of these lines; and by Prop. XI. set off upon each of them from P, the given distance of the circle from its pole, as PL, Pλ, PM, Pμ, &c. and thro' all the points L, M, D, O, R, &c. draw a curve line, for the circle required. 49.

Or thus.

Draw the line of measures PCG, and by Prop. XI. make CG = the distance of the parallel great circle from the pole of projection, and draw AGK perpendicular to it, which will represent a great circle whose pole is P. Draw any number of right lines thro' P to AK, as AP, BP, HP, &c. and by Prop. XI. set off from AK the parts AL, BM, HO, &c. each equal to the circle's distance from its parallel great circle. Then all the points L, M, D, O, &c. being joined by a regular curve, will represent the parallel circle required. 49.

Or thus.

Thro' the center of projection C draw the line of measures DCF, and the radius of projection CW perpendicular to it, and AGK $\perp$ GC, for a great circle whose pole is P. Draw wp = WP, and wa $\perp$ to it, draw any number of right lines, AP, BP, GP, &c. and make pg, pb, pa, &c. = PG, PB, PA, 49.

Fig. &c. alſo make the $<$ *pwl* and *pwx* = the circle's
49. diſtance from its pole P (or *awl* = the diſtance from its parallel great circle); and upon PG, PB, PA, &c. make PD, PM, PL, &c. = *pd*, *pm*, *pl*, &c. reſpectively.

Or make GD, BM, AL, &c. = *gd*, *bm*, *al*, &c. After the ſame manner, find the points O, R, &c. and thro' all the points R, O, D, M, L, &c. draw a regular curve, making no angles, which will repreſent the parallel required. Likewiſe where any line *ap* cuts *wx*, that diſtance from *p* will give the point λ, or is = Pλ; and ſo of any other of the lines *bp*, *gp*, &c.

The reaſon of this proceſs will be plain, if you ſuppoſe the points p, w applied to P, W; *and g, b, a, &c. ſucceſſively to* G, B, A, &c. *for then d, m, l, will fall upon* D, M, L, &c.

By the Scale.

45. Take the tangents of the circle's neareſt and furtheſt diſtance from the pole of projection, and ſet from C to *f* and *d*, gives the vertices, and biſect *df* in H; then take half the difference, or half the ſum, of the ſecants of the greateſt and leaſt diſtances from the pole of projection, and ſet from H, to K or *k* for the focus of the ellipſis or hyperbola, which may then be deſcribed.

49. Cor. *If the curve be required to paſs thro' a given point* S; *meaſure* PS *by Prop.* XI, *and then the curve may be drawn by this Problem.*

P R O P. XVI. *Prob.*

47. *To find the pole of any circle in the projection,* DMF.

Rule.

From the center of projection C, draw the radius of projection CA perpendicular to the line of meaſures

sures DF. And to A the projecting point, draw DA, FA, and bisect the angle DAF by the line AP, then P is the pole. But if the curve be an hyperbola, as *fm*, *fig.* 45, you must produce *d*A, and bisect the angle *f*AG. And in a parabola, where the point *d* is at an infinite distance, bisect the angle *f*AE. Fig. 47.

Or thus; Drawing CA perpendicular to DC, draw DA, and make the angle DAP = the circle's distance from its pole, gives the pole P.

By the Scale.

Draw the radius of projection CA ⊥ to the line of measures DF. Apply CD CF to the tangents, and set the tangent of half the difference of their degrees from C to P, if D, F lye on contrary sides of C; but half the sum if on the same side, gives P the pole.

Or thus; By Prop. XI. set off from D to P, the circle's distance from its pole, gives the pole P.

Cor. *If it be a great circle as* BG; *draw the line of measures* GC, *and* CA ⊥ *to it, and equal to the radius of projection*; *make* GAP *a right angle, and* P *is the pole.* 46.

PROP. XVII. *Prob.*

To measure any arch of a lesser circle; *or to set any number of degrees thereon.*

Rule.

Let F*n* be the given circle. From the center of projection C, draw CA perpendicular to the line of measures GH. To P the pole of the given circle draw AP, and AO bisecting the angle CAP. And draw AD perpendicular to AO. Describe the circle G*l*H (by Prop. XIII.) as far from the pole of projection C, as the given circle is from its pole P. And thro' any given point *n* in the circle F*n*, 46.

Fig. draw D*nl*, gives H*l* the number of degrees = F*n*.
45. Or the degrees being given and set from H to *l*, the line D*l* cuts off F*n* equal thereto.

Or thus; AO being drawn as before, erect OS perpendicular to CO; thro' the given point *n* draw P*n* cutting OS in Q, then thro' Q draw C*l*, and the angle QCP is = F*n*. Or making QCP = the degrees given, draw PQ*n*, and arch F*n* = these degrees.

Or thus; AO, AP, being drawn as before, draw AG perpendicular to AP, and GB perpendicular to GC. Thro' the given point *n* draw PB cutting GB in B, and draw OB, then the < GOB = arch F*n*. Or making < GOB = the given degrees; draw PB, and it cuts off F*n* = the degrees given.

By the Scale.

Let C be the center of projection, P the pole of the given circle. Apply CP to the tangents, and set the tangent of its half from C to O, and the co-tangent of its half from C to D; with radius CG = tangent of the degrees in FP the given circle's distance from its pole, describe the circle GSH. Then D*l* drawn thro' *n* or *l*, cuts off H*l* = F*n*.

Or thus; O being found as before, erect OS perpendicular to CO; thro' the given point *n* draw PQ*n*, and < QCH = F*n*.

Or thus; Apply CP to the tangents, and set the co-tangent thereof from C to G. Erect GB perpendicular to GC. Thro' *n* draw P*n*B, and draw BO; then < GOB = F*n*.

48. Cor. *If the lesser circle be perpendicular to the plain of projection as* VHK. *You have no more to do but to draw the perpendiculars* VC, HG, *to its parallel great circle* CI. *Then* CG (*measured by Prop.* XI.) *will be equal to* VH; *or the degrees set from* C *to* G, *cuts off* VH *equal thereto.*

SCHO-

Fig.

SCHOLIUM.

This sort of projection is little used, by reason of 48.
several of the circles of the sphere fall in ellipses and hyperbolas, which are very difficult to describe. Notwithstanding it is very convenient for solving some Problems of the sphere, because all the great circles are projected into right lines. And this sort, or the Gnomonic Projection is the very foundation of all dialling. For if the sphere be projected on any plane, and upon that side of it on which the sun is to shine; and the projected pole be made the center of the dial, and the axis of the globe the Stile or Gnomon, and the radius of projection its height; you will have a dial drawn with all its furniture. Upon this account it deserves to be more taken notice of, than at present it is. I have in the foregoing propositions given, I think, all the fundamental principles of this kind of projection, having met with little or nothing done upon this subject before.

GENERAL PROBLEM.

To project the sphere upon any given plane.

Before you can project the sphere upon any plane, you must have a perfect knowledge of all its circles, and their positions in respect of one another; the distances of the lesser circles from their poles, and from their parallel great circles; the angles made by great circles, or their inclinations, to one another, particularly to the primitive circle, on whose plane (or a parallel thereto) you are about to project the sphere. Then after the primitive circle is described; you must describe all other circles concerned in the Problem, according to the rules of that sort of Projection, you are going to use;

Fig. and the interſection of theſe circles will determine the Problem.

And note, that the Projection of the concave ſide of the ſphere is more fit for aſtronomical purpoſes; for in looking at the heavens, we view the concavity. But it is better to project the convex hemiſphere in geography, becauſe we ſee the convex ſide only.

The principal Points, Angles and Circles of the Sphere are as follows.

I. *Points.*

52. 1. *Zenith* is the point over our heads, Z.

53. 2. *Nadir* is the point under our feet, N.

55. 3. *Poles* of the world are 2 points, round which the diurnal revolution is performed, P the north pole, *p* the ſouth pole. A line drawn through the poles, is called the *Axis* of the world, as P*p*.

4. *The Center* of the earth or of the heavens, C.

5. *Equinoctial Points*, are the points of interſection of the Equator and Ecliptic, ♈, ♎.

6. *Solſtitial Points*, are the beginning of Cancer and Capricorn, ♋, ♑.

II. *Great Circles.*

1. *Equinoctial*, is a circle 90 degrees diſtant from the poles of the world, as EQ.

2. *Meridians*, or *hour Circles*; are circles paſſing thro' the poles of the world, as P ☉ *p*, PE*p*, &c.

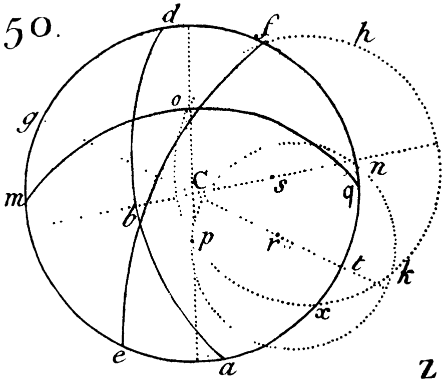
3. *Solſtitial Colure*, is a meridian paſſing thro' the ſolſtitial points, as P♋*p*.

4. *Equinoctial Colure*, is a meridian paſſing thro' the equinoctial points, P C*p*.

5. *Ecliptic* is the circle thro' which the ſun ſeems to move in a year, ♋ ♑; it cuts the equinoctial at an angle of 23° 30′, in paſſing thro' the equinoctial points. In this are reckoned the 12 Sines, ♈, ♉, ♊, ♋, ♌, ♍, ♎, ♏, ♐, ♑, ♒, ♓.

6. *Ho-*

50.



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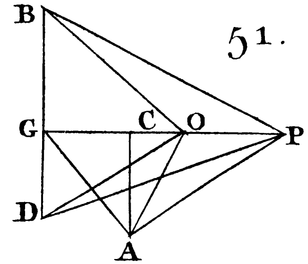

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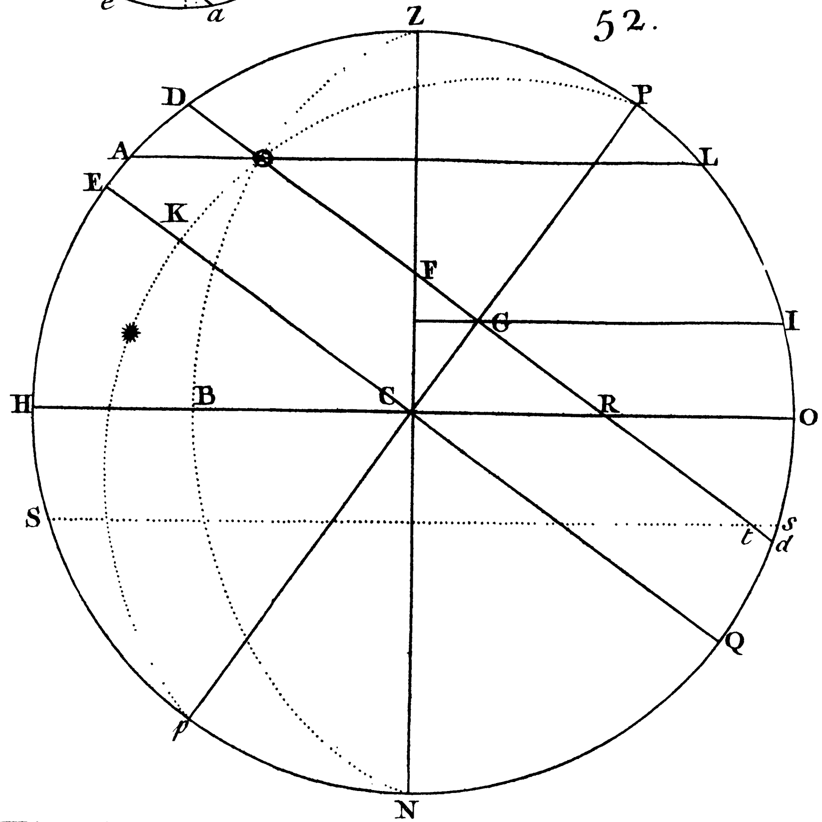

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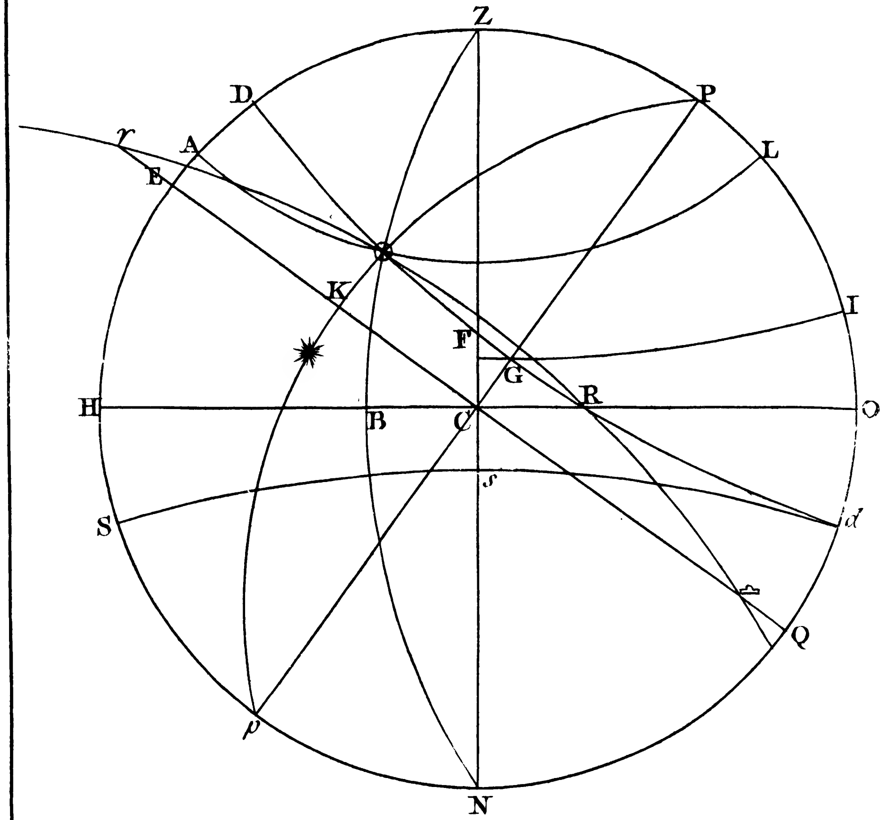


Projection. X. p. 50.

6. *Horizon*, is a circle dividing the upper from the lower hemisphere, as HO, being 90° distant from the Zenith and Nadir. Fig. 52. 53.

7. *Vertical Circles*, are circles passing thro' the Zenith and Nadir, Z ☉ N. 55.

8. *Circles of Longitude* in the heavens, pass thro' the poles of the ecliptic and cut it at right angles.

9. *Meridian of a Place*, is that Meridian which passes thro' the Zenith, as PZH.

10. *Prime Vertical*, is that which passes thro' the east and west points of the horizon.

III. *Lesser circles.*

1. *Parallels of Latitude* are parallel to the equinoctial on the earth, *parallels of altitude* are parallel to the horizon, *parallels of declination* are parallel to the equinoctial in the heavens.

2. *Tropics*, are 2 circles distant 23° 30′ from the equinoctial, the tropic of *Cancer* towards the north, the tropic of *Capricorn* towards the south.

3. *Polar Circles*, are distant 23° 30′ from the poles of the world, the *Arctic* circle towards the north, the *Antarctic* towards the south.

IV. *Angles and Arches of Circles.*

1. *Sun's* (or *Star's*) *Altitude*, is an arch of the Azimuth between the sun and horizon, as ☉ B.

2. *Amplitude* is an arch of the horizon, between sun-rising and the east, or sun-setting and the west.

3. *Azimuth*, is an arch of the horizon between the sun's Azimuth circle, and the north or south, as HB, or OB; or it is the angle at the zenith, HZB or OZB.

4. *Right Ascension* is an arch of the equator between the sun's meridian, and the first point of Aries, as ♈ K.

5. *Ascensional Difference* is an arch of the equinoctial, between the sun's meridian, and that point

Fig. of the equinoctial that rises with him, or it is the
52. angle at the pole between the sun's and the six o'clock
53. meridian.
55. 6. *Oblique Ascension* or *Descension*, is the sum or difference of the right ascension and the ascensional difference.

7. *Sun's Longitude*, is an arch of the ecliptic, between the sun and first part of Aries, as ♈ ☉.

8. *Declination* is an arch of the meridian, between the equinoctial and the sun, as ☉K.

9. *Latitude of a Star*, is an arch of a circle of longitude between the star and ecliptic.

10. *Latitude of a Plane*, in an arch of the meridian between the equinoctial and the place.

11. *Longitude* of a place on the earth is an arch of the equinoctial, between the first meridian (Isle of *Ferro*), and the meridian of the place. And *diff. longitude*, is an arch of the equator, between the meridians of the two places, or the angle at the pole.

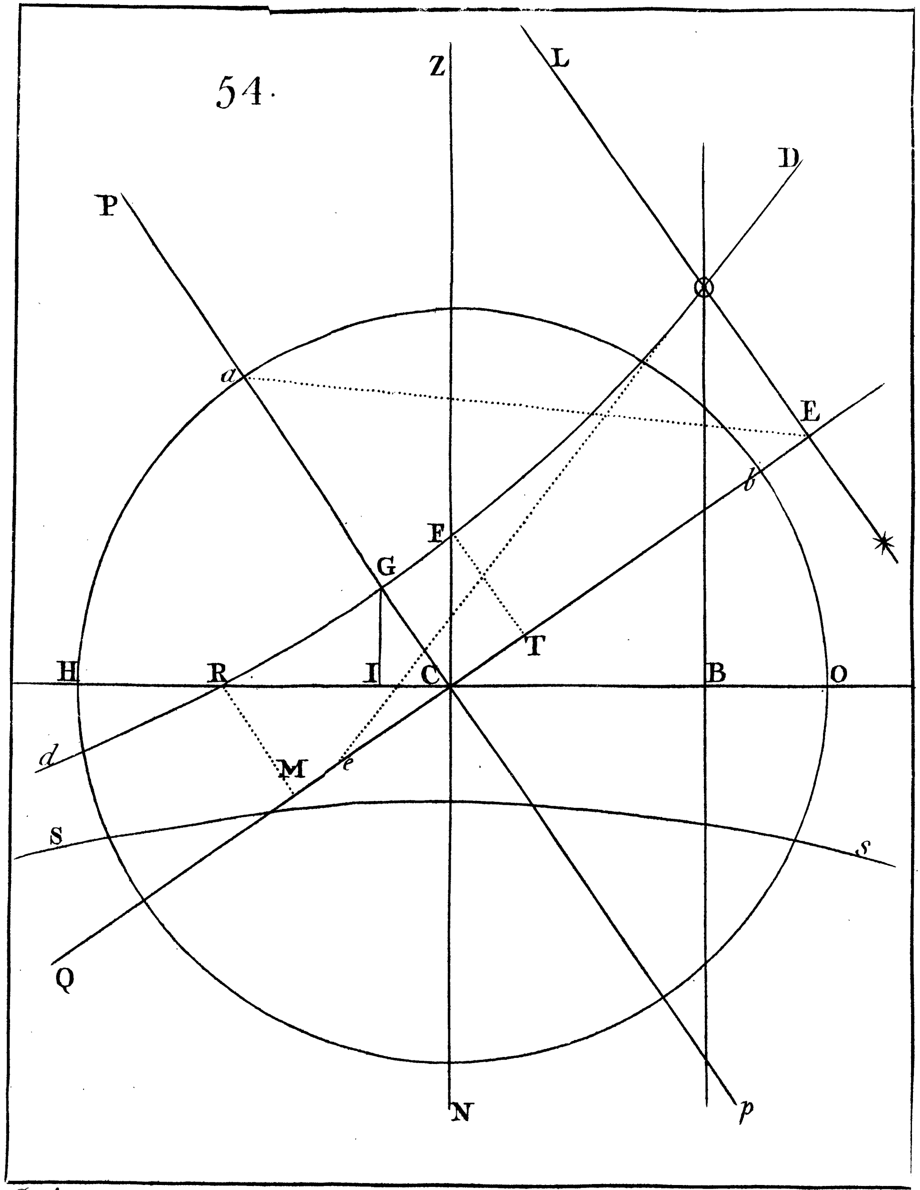
12. *Hour of the Day*, is an arch of the equinoctial, between the meridian of the place and the sun's meridian, as EK; or it is the angle they make at the pole, as EPO.

Example I.

To project the sphere upon the plane of the meridian, for May 12, 1767. *Latitude* 54° ½ *north, at a quarter past* 9 *o'clock before noon.*

I. *By the Orthographic Projection.*

52. Here we will project the convex side of the eastern hemisphere. With the chord of 60° degrees describe the primitive circle or meridian of the place HZON. Thro' the center C draw the horizon HO; set the latitude 54½ from O to P and from H to *p*, and draw P*p* the 6 o'clock meridian. Thro' C draw EQ perpendicular to P*p* for the equinoctial. Make ED, Q*d* 18° 5′ the declination May 12, and draw D*d* the sun's parallel for that day. By Prop.



Projection. XI. p. 52.

Prop. XI. make ☉ G (3 ¼ hours or) 48° 45′ the sun's distance from the hour of 6, then ☉ is the sun's place. Thro' ☉ by Prop. V. draw AL parallel to H ☉ for the sun's parallel of altitude. By Prop. VII. draw the meridian P ☉ *p* and the azimuth Z ☉ N. Also the ecliptic will be an ellipsis passing thro' ☉, which cannot conveniently be drawn in this projection. Also draw the parallel S*s* 18° below the horizon, and where it intersects D*d* is the point of day break, if there is any. Now the sun is at *d* at 12 o'clock at night, and rises at R, at 6 o'clock is at G, due east at F, at ☉ a quarter past 9, and is at D in the meridian at 12 o'clock. Fig. 52.

Draw GI parallel to HO. Then GR measured by Prop. X. is 27° 14′, and turned into time (allowing 15 degrees for an hour) shows how long the sun rises before 6, to be $1^h\ 49^m$; GI measured by Prop. X. gives the azimuth at 6, 79° 16′. CR by Cor. Prop. X. gives the amplitude 32° 19′, and CF gives his altitude when east 22° 25′. FG 13° 28′ (turned into time) is 54^m, and shews how long after 6 he is due east. IO is his altitude at 6, 14° 38′. AH 41° 53′ is his altitude at ☉, or a quarter past 9; and ☉L measured by Prop. X. is his azimuth from the north at the same time, 122° 40′. And thus the place of the moon or a star being given, it may be put into the projection, as at ✱. And its altitude, azimuth, amplitude, time of rising, &c. may all be found, as before for the sun.

II. *Stereographically.*

To project the sphere on the plane of the meridian, the projecting point in the western point of the horizon; with cord of 60, draw the primitive circle HZON, and thro' C draw HO for the horizon, and ZN perpendicular thereto for the prime vertical. Set the latitude from O to P, and from H to *p*, and draw P*p* the 6 o'clock meridian, and 53.

EQ

Fig. EQ perpendicular thereto for the equinoctial.
53. Make ED, Q*d* the declination, and by Prop. XII. draw DG*d*, the sun's parallel for the day. Draw the meridian P ☉ *p* by Prop. XVII. making an angle of 41° 15′ with the primitive, to intersect the sun's parallel in ☉, the sun's place at $9^h \frac{1}{4}$. Thro' ☉, by Prop. XII. draw the parallel of altitude A☉L; thro' ☉ draw, by Prop. XVII. the azimuth Z☉N. And by Prop. XII. draw the parallel S*sd* 18° below the horizon, if it cut R*d*, gives the point of day break. And thro' G draw the parallel of altitude GI. Lastly, by Prop. XX. thro' ☉ draw the great circle ♈☉♎ cutting the equinoctial EQ at an angle of 23°: 30′, and this is the ecliptic, ♈ the first point of Aries, and ♎ that of Libra.

This done, *d*R measured by Prop. XXIII. is 62° 46′, shows the time of sun rising; CR by Prop. XXII. is the amplitude 32° 19′. GI 79° 16′ by Prop. XXIII. the sun's azimuth at 6. IO 14° 38′ his altitude at 6. CF 22° 25′ by Prop. XXII. his altitude when east. GF 13° 28′ the time when he is due east. ☉B 41° 53′ by Prop. XXII. his altitude at a quarter past 9; the ∠☉ZP 122° 40′ by Prop. XXIV. his azimuth at that time. Also ♈☉, by Prop. XXII. is his longitude 51° 7′. ♈K his right ascension, 48° 40′.

And the place of the moon or a star being given, it may be put into the scheme as at *; and its time of rising, amplitude, azimuth, &c. found as before.

III. *Gnomonically.*

54. To project the eastern hemisphere upon a plane parallel to the meridian. About the center of projection C describe the circle HON with the tangent of 45 the radius of projection, for the primitive. Thro' C draw the horizon HO, and the prime vertical

tical ZN perpendicular thereto. Set the latitude $54\frac{1}{2}$ from H to *a*, and draw the 6 o'clock meridian P*p*, and the equinoctial EQ perpendicular to it. Set the tangent of 48° 45′ (equal to $3\frac{1}{4}$ hours) from C to E, and by Prop. X. draw the meridian EL parallel to P*p*. Make E*e* = E*a*, and < E*e* ⊙ = 18° 5′ the sun's declination, then by Prop. XI. ⊙ is the sun's place. Thro' ⊙ draw the hyperbola D⊙*d* (by Prop. XIV.) for the sun's parallel of declination; and draw ⊙B perpendicular to HO, for his azimuth circle. And draw GI perpendicular to HO, and RM, FT, ∥ P*p*. Also the ecliptic is a right line passing thro' ⊙, and cutting EQ at an angle of 23° 30′, which is difficult to draw in this projection. Fig. 54.

Also by Prop. XIV. Draw the parallel S*s* 18° below the horizon, and if it intersects D*d*, it gives the point of sun rise.

Then if by Prop. XVII. or XI. you measure GR or rather CM, 27° 14′, you have the time of sun rising; GF or CT 13° 28, the time when he is due east. Also by Prop. XI. if you measure CR you have the amplitude 32° 19′. CI the comp. of his azimuth at six, 10° 44′. IG by Prop. XII. his altitude at 6, 14° 38′. CF his altitude when east, 22° 25′. And by Prop. XI. ⊙B = 41° 53′, his altitude a quarter past 9. CB the complement of his azimuth at that time, 32° 40′.

And the place of the moon or a star being given, its place in the projection may be determined as before, and all the requisites found.

Ex. 2.

To project the sphere upon the plane of the solstitial colure for latitude $54\frac{1}{2}$ N. May 23, 1767, at 10 o'clock in the morning.

Stereogra-

Fig.

Stereogrophically.

55. The projection of the western hemisphere, the first point of Libra, the projecting point. Describe the solstitial colure PEpQ, and the equinoctial colure Pp perpendicular to it; and thro' C draw the equinoctial EQ perpendicular to Pp. Set 23° 30 from E to ♋, and from Q to ♑, and draw the ecliptic ♋ ♑. Set the sun's longitude 61° 42′ from C to ☉, and thro' ☉ draw P☉Kp for the 10 o'clock meridian. Make KA (two hours or) 30°, and draw PAp for the meridian of the place. Set the latitude of the place $54\frac{1}{2}$ from A to Z, and Z is the zenith. About the pole Z describe the great circle BHS for the horizon of the place. Thro' Z and ☉ draw an azimuth circle Z☉B.

Then you have ☉K the sun's declination 20° 33′. CK his right ascension 59° 35′. ☉B his altitude at 10 o'clock 49° 10′; the ∠ AZ☉ or PZ☉ his azimuth at 10 = HB, 45° 44′. H the south point of the horizon. I the point of the ecliptic that is in the meridian. T the point of the ecliptic that is setting in the horizon.

Example. 3.

To project the sphere on the plane of the horizon, Lat. $35\frac{1}{2}$, *N. July* 31, 1767, *at* 10 *o'clock.*

Gnomonically.

56. To project the upper hemisphere on a plane parallel to the horizon. With the radius of projection and center C, describe the primitive circle ADB. Thro' C draw the meridian PE, and AS perpendicular to it for the prime vertical. Set off CP $35\frac{1}{2}$ the latitude and P is the N. Pole, and perpendicular to CP draw Pp the 6 o'clock meridian. Set the complement of the latitude from C to E; and draw EQ perpendicular to CE for the equinoc-

equinoctial. Make EB 30° (or 2 hours) and draw the 10 o'clock meridian PB. Set the sun's declination 18° 27′ from B to ⊙. And ⊙ is the place of the sun at 10 o'clock. Thro' ⊙ draw the azimuth circle CQ; likewise thro' ⊙, a parallel to the equinoctial EQ may easily be described by Prop. XV. for the sun's parallel that day. Fig. 56.

Then C ⊙ measured by Prop. XI. is 31° 30′ the complement of the altitude. And the angle EC⊙ measured by Cor. Prop. XII. is his azimuth, 65° 10.

SCHOLIUM.

After this manner may any Problems of the Sphere be solved by any of these Projections, or upon any planes, but upon some more commodiously than upon others. And if in a spherical triangle any sides or angles be required, they may be projected according to what is given therein, according to any of these kinds of projection before delivered; and it will be most easily done, when you chuse such a plane to project on, that some given side may be in the primitive, or a given angle at the center; and then you need draw no more lines or circles than what are immediately concerned in that Problem. But always chuse such a plane to project on, where the lines and circles are most easily drawn, and so that none of them run out of the scheme.

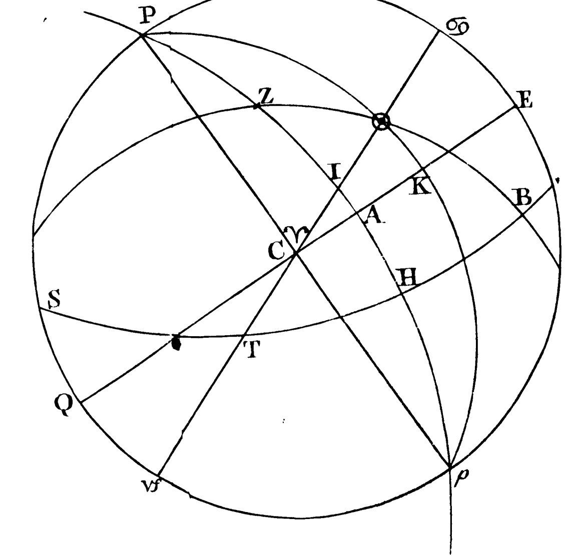
F I N I S.

ERRATA.

b ſignifies reckon from the bottom.

pag	line	read
5	1	fig. 2.
8	18	fig. 5.
9	3b	off the ſines,
13	17	fig. 12.
14	2b	the 2 laſt lines ſhould be indented and roman.
	21	$CpA + CAp$;
15	3	the 3d line ſhould be indented, and the 3 following lines roman.
16	5	$ECI = CAD$
18	4	if, p, q, be
20	7	projection C,
21	15	$\frac{OG + OD}{2}$:
25	9b	points A, B, G,
29	6b	interſection p,
30	2	gCt required.
	11b	pole p, draw
33	17	of this circle
36	6b	TV, and
	2b	at s cut
38	2b	to oS;
43	2	$CL =$ radius
44	20	A to d,

55



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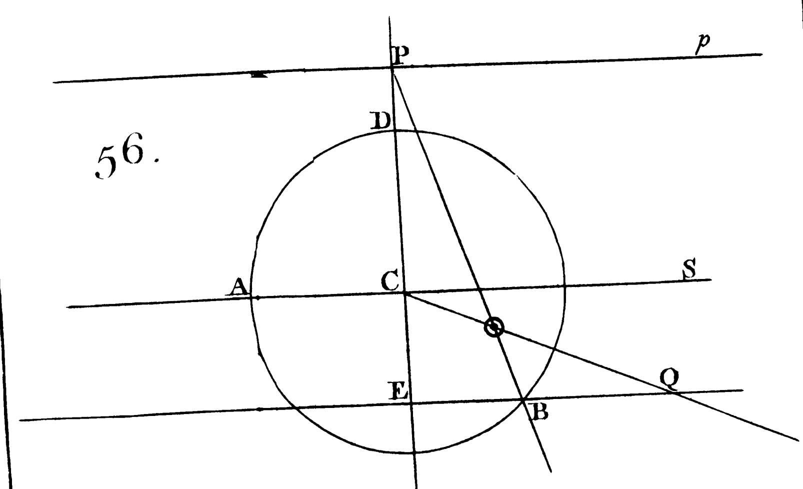


Projection.

XII. p. last.

THE LAWS OF Centripetal and Centrifugal FORCE.

SHEWING,

The MOTION of BODIES in Circular Orbits, and in the Conic Sections, and other Curves.

And explaining the perturbating Force of a third Body. With many other Things of like Nature.

Being a Work preparatory to ASTRONOMY, and the very Basis thereof. And absolutely necessary to be known by all such as desire to be Proficients in that SCIENCE.

Solis uti varios cursus, lunæque meatus
Noscere possemus, quæ vis, & causa cieret.

LUCRET. Lib. V.

THE

PREFACE.

IN the following Treatise, I have explained and demonstrated the Laws of Centripetel Forces; a doctrine upon which all Astronomy is grounded; and without the knowledge of which, no rational account can be given of the motions of any of the celestial bodies, as the Comets, the Planets, and their Satellites. From these laws are derived the causes of the various seeming irregularities observed in their motions; such as their accelerations and retardations, their approaching to, and receding from the center of force; irregularity, only in appearance; but in reality, these motions are truly regular and conformable to the established laws of Nature. From this foundation we trace the way or path of all the planets, and discover the origin and spring of all the celestial motions, and clearly understand and account for all the phænomena thence arising.

In the first section, you have the Centripetal Forces of bodies revolving in circles; their velocities, periodic times, and distances compared together; their relations and proportions to each other; and that when they either revolve about the same center, or about different ones. The different motions caused by different forces, or by different central attracting bodies, are here shewn. We have given likewise the periodic time of a simple pendulum revolving with a conical motion; and also the center of Turbination, and the periodic time of a compound pendulum, or a system of bodies, revolving with a conical motion; as properly belonging to the doctrine of Centripetal Forces.

In the second section we have shewn the motion of bodies in the Ellipsis, Hyperbola, and Parabola; and in other Curves. The proportion of the Centripetal Forces, and velocities in different parts of the same Curve. The law of Centripetal Force to describe a given Curve, and the velo-

city in any point of it; and more particularly with respect to that law of Centripetal Force that is reciprocally as the square of the distance; which is the grand law of Nature in regard to the action of bodies upon one another at a distance; and according to this law, is shewn the motion of bodies round one another, and round their common center of gravity, and the orbits they will describe.

In the third section we have given the disturbing or perturbating force of a third body, acting upon two others that revolve round one another. From these principles are deduced the errors caused in the motion of a Satellite, moving round its primary planet. Towards the end, are several propositions, by means whereof, the motion of the Nodes, and variation of inclination of a Satellite's orbit, and such like things may be computed. As these things are all laid down for the sake of understanding our own System, I have inserted some few things, by way of illustration of the rules, in regard to the Moon and Jupiter. But as to the Moon, there are some things so very intricate, and require such long and tedious calculations, as would require a volume of themselves; so that the small room I am confined to cannot admit of them; and few would trouble themselves to read them, if they were there. This last section concludes with a few things of another kind, but depending on the principles of Centripetal Forces.

Several of these things about Centripetal Forces are calculated by the method of Fluxions; and cannot easily be done any other way; and most of them taken from my book of Fluxions. And several other things relating to Centripetal Forces, you will also find in that book; being sorry to trouble the reader too much with repeating what I have written and published elsewhere.

W. Emerson.

THE LAWS OF Centripetal and Certrifugal Force.

DEFINITIONS.

DEF. I.

THE center of attraction, is the point towards which any body is attracted or impelled.

DEF. II.

Centripetal force, is that force by which a body is impelled to a certain point, as a center. Here all the particles of the body are equally acted on by the force.

DEF. III.

Centrifugal force, is the resistance a moving body makes to prevent its being turned out of its direct course. This is opposite and equal to the centripetal force; for action and re-action are equal and contrary.

DEF. IV.

Angular velocity, is the quantity of the angle a body describes in a given time, about a certain point, as a center. *Apparent velocity* is the same thing.

DEF. V.

Periodical time, is the time of revolution of a body round a center.

SECT. I.

The motion of bodies in Circular ORBITS.

PROP. I.

Fig. *The centripetal forces, whereby equal bodies at equal distances from the centers of force, are drawn towards these centers; are as the quantities of matter in the central bodies.*

For since all attraction is made towards bodies, every part of the attracting body must contribute its share in that effect. Therefore a body twice as great will attract the same body twice as much; and one thrice as great, thrice as much, and so on. Therefore the attraction of the central body; that is, the centripetal force, is as the quantity of matter in the attracting or central body.

Cor. 1. *Any body whether great or little, placed at the same distance, is attracted thro' equal spaces in the same time, by the central body.*

For tho' a body twice or thrice as great as another, is drawn with twice or thrice the force; yet it will acquire no greater velocity, nor pass thro' a greater space. For (Prop. V. Cor. 2. Mechan.) the velocity generated in a given time, is as the force directly, and quantity of matter reciprocally; and the force, which is the weight of the body, being as the quantity of matter; therefore the velocity generated is as the quantity of matter directly, and quantity of matter reciprocally, and therefore is a given quantity.

Cor.

Cor. 2. *Therefore the centripetal force, or force towards the center, is not to be measured by the quantity of the falling body; but by the space it falls thro' in a given time. And therefore it is sometimes called an accelerative force.* Fig.

PROP. II.

If a body revolves in a circle, and is retained in it, by a centripetal force, tending to the center of it; put 1.
R = *radius of the circle or orbit described,* AC.
F = *absolute force, at the distance* R.
s = *the space, a falling body could descend thro', by the force at* A, *and*
t = *time of the descent.*
$\pi = 3.1416$.
Then its periodic time, or the time of one revolution will be $\pi t\sqrt{\frac{2R}{s}}$.

And the velocity, or space it describes in the time t, *will be* $\sqrt{2Rs}$.

For let AB be a tangent to the circle at A; take AF an infinitely small arch, and draw FB perp. to AB, and FD perp. to the radius AC. Let the body descend thro' the infinitely small hight AD or BF, by the centripetal force in the time 1. Now that the body may be kept in the circular orbit AFE, it ought to describe the arch AF in the same time 1. The circumference of the circle AE is $2\pi R$, and the arch AF $= \sqrt{2R \times AD}$.

By the laws of falling bodies $\sqrt{s} : t :: \sqrt{AD} : t\sqrt{\frac{AD}{s}}$ = time of moving thro' AD or AF. And by uniform motion, as AF, to the time of its description :: circumference AFEA, to the time of one

Fig.
1. one revolution; that is, $\sqrt{2R \times AD} : t\sqrt{\frac{AD}{s}}$: :
$2\pi R$: periodic time $= \frac{2t\pi R}{\sqrt{2Rs}} = \pi t\sqrt{\frac{2R}{s}}$.

Alſo by the laws of uniform motion, $t\sqrt{\frac{AD}{s}}$ or time of deſcribing AF : AF or $\sqrt{2R \times AD}$: : $t : \sqrt{2Rs} =$ the velocity of the body, or ſpace deſcribed in time t.

Cor. 1. *The velocity of the revolving body, is equal to that which a falling body acquires in deſcending thro' half the radius* AC, *by the force at* A *uniformly continued.*

For $\sqrt{s}$ (hight) : $2s$ (the velocity) : : $\sqrt{\frac{1}{2}R}$ (the hight) : $\sqrt{2Rs}$, the velocity acquired by falling thro' $\frac{1}{2}$ R.

Cor. 2. *Hence, if a body revolves uniformly in a circle, by means of a given centripetal force; the arch which it deſcribes in any time, is a mean proportional between the diameter of the circle, and the ſpace which the body would deſcend thro' in the ſame time, and with the ſame given force.*

For 2R (diameter) : $\sqrt{2Rs}$: : $\sqrt{2Rs}$: s; where $\sqrt{2Rs}$ is the arch deſcribed, and s the ſpace deſcended thro', in the time t.

2. Cor. 3. *If a body revolves in any curve* AFQ, *about the center of force* s; *and if* AC *or* R *be the radius of curvature in any point* A; $s =$ *ſpace deſcended by the force directed to* C. *Then the velocity in* A *will be* $\sqrt{2Rs}$.

For this is the velocity in the circle; and therefore in the curve, which coincides with it.

PROP.

PROP. III.

If several bodies revolve in circles round the same or different centers; the periodic times will be as the square roots of the radii directly, and the square roots of the centripetal forces reciprocally. 1.

Let F = centripetal force at A tending to the center C of the circle.
V = velocity of the body.
R = radius AC of the circle.
P = periodic time.

Then (Prop. II.) $P = \pi t \sqrt{\frac{2R}{s}}$. But s is as the force F that generates it; whence $P = \pi t \sqrt{\frac{2R}{F}}$, and since 2, π and t are given quantities, therefore $P \propto \sqrt{\frac{R}{F}}$.

Cor. 1. *The periodic times are as the radii directly, and the velocities reciprocally.*

For (Prop. II.) $V = \sqrt{2Rs} = \sqrt{2RF}$, and $V^2 = 2RF$, and $P = \pi t \sqrt{\frac{2R}{F}}$, and $PP = \pi^2 t^2 \times \frac{2R}{F}$, therefore $P^2 V^2 = \pi^2 t^2 \times 4R^2$, and $P^2 = \frac{\pi^2 t^2 \times 4R^2}{V^2}$, and $P = \frac{\pi t \times 2R}{V} \propto \frac{R}{V}$.

Cor. 2. *The periodic times are as the velocities directly, and the centripetal forces reciprocally.*

For $V^2 = 2Rs = 2RF$; and $R = \frac{VV}{2F}$, and $\frac{R}{V} = \frac{V}{2F} \propto \frac{V}{F}$. But (Cor. 1.) $P \propto \frac{R}{V} \propto \frac{V}{F}$.

 Cor.

Fig. 1. Cor. 3. *If the periodic times are equal; the velocities, and also the centripetal forces, will be as the radii.*

For if P be given; then $\frac{R}{F}$, and $\frac{R}{V}$, and $\frac{V}{F}$ are all given ratios.

Cor. 4. *If the periodic times are as the square roots of the radii; the velocities will be as the square roots of the radii, and the centripetal forces equal.*

For (Prop. III. and Cor. 1.) putting $\sqrt{R}$ for P, we have $\sqrt{R} \propto \sqrt{\frac{R}{F}} \propto \frac{R}{V}$. Therefore $1 \propto \frac{1}{\sqrt{F}} \propto \frac{\sqrt{R}}{V}$, and $\sqrt{R} \propto V$, and $\sqrt{F}$ is a given quantity.

Cor. 5. *If the periodic times are as the radii; the velocities will be equal, and the centripetal forces reciprocally as the radii.*

For putting R for P, we have $R \propto \sqrt{\frac{R}{F}} \propto \frac{R}{V}$; whence $\sqrt{R} \propto \frac{1}{\sqrt{F}}$, and $1 \propto \frac{1}{V}$; that is, $R \propto \frac{1}{F}$, or the centripetal force is reciprocally as the radius; and V is a given quantity.

Cor. 6. *If the periodic times are in the sesquiplicate ratio of the radii; the velocities will be reciprocally as the square roots of the radii, and the centripetal forces reciprocally as the squares of the radii.*

Put $R^{\frac{3}{2}}$ for P, then $R^{\frac{3}{2}} \propto \sqrt{\frac{R}{F}} \propto \frac{R}{V}$; and $R \propto \frac{1}{\sqrt{F}}$ or $RR \propto \frac{1}{F}$, and $\sqrt{R} \propto \frac{1}{V}$.

Cor. 7. *If the periodic times be as the nth power of the radius; then the velocities will be reciprocally as the $n - 1^{th}$ power of the radii, and the centripetal forces*

forces reciprocally as the $2n - 1^{th}$ *power of the radii.* Fig. 1.

Put R^n for P, then $R^n \propto \sqrt{\frac{R}{F}} \propto \frac{R}{V}$. Whence $R^{2n} \propto \frac{R}{F}$, and $R^{2n-1} \propto \frac{1}{F}$. Alfo $R^{n-1} \propto \frac{1}{V}$.

PROP. IV.

If feveral bodies revolve in circles round the fame or different centers; the velocities are as the radii directly, and periodic times reciprocally. 1.

For putting the fame letters as in Prop. III. we have (by Prop. II.) $V = \sqrt{2Rs} = \sqrt{2RF}$; and $P \propto \frac{V}{F}$ (by Cor. 2. Pr. III.), and $PF \propto V$, and $F \propto \frac{V}{P}$. Whence $V = \sqrt{2RF} = \sqrt{2R \times \frac{V}{P}}$, and $V^2 = \frac{2RV}{P}$, and $V = \frac{2R}{P} \propto \frac{R}{P}$.

Cor. 1. *The velocities are as the periodical times, and the centripetal forces.*

For we had $PF \propto V$.

Cor. 2. *The fquares of the velocities are as the radii and the centripetal forces.*

For $V = \sqrt{2RF}$.

Cor. 3. *If the velocities are equal; the periodic times are as the radii, and the radii reciprocally as the centripetal forces.*

For if V be given, its equal $\frac{R}{P}$ is a given ratio; and $\sqrt{RF}$ is given, whence $R \propto \frac{1}{F}$.

Fig. 1. Cor. 4. *If the velocities be as the radii, the periodic times will be the same; and the centripetal forces as the radii.*

For then V or R $\propto \frac{R}{P}$, and $1 \propto \frac{1}{P}$. Also R $= \sqrt{2RF}$, whence R $\propto$ F.

Cor. 5. *If the velocities be reciprocally as the radii; the centripetal forces are reciprocally as the cubes of the radii; and the periodic times as the squares of the radii.*

For put $\frac{1}{R}$ for V, then (Cor. 2.) $\frac{1}{R} = \sqrt{2RF}$, $\frac{1}{RR} = 2RF$, whence F $\propto \frac{1}{R^3}$. Also $\frac{1}{R} \propto \frac{R}{P}$, and P $\propto$ RR.

PROP. V.

1. *If several bodies revolve in circles about the same or different centers; the centripetal forces are as the radii directly, and the squares of the periodic times reciprocally.*

Put the same letters as in Prop. III. Then (Prop. II.) $P = \pi t\sqrt{\frac{2R}{s}} = \pi t\sqrt{\frac{2R}{F}}$, and $PP = \pi\pi tt \times \frac{2R}{F}$, and $PPF = 2\pi\pi ttR$; whence $F = \frac{2\pi\pi ttR}{PP} \propto \frac{R}{PP}$.

Cor. 1. *The centripetal forces are as the velocities directly, and the periodic times reciprocally.*

For (Prop. IV.) V $\propto \frac{R}{P}$, and F $\propto \frac{R}{PP} \propto \frac{V}{P}$.

Cor. 2. *The centripetal forces, are as the squares of the velocities directly, and the radii reciprocally.*

For

For (Cor. 1.) F $\propto \frac{V}{P}$, and FP $\propto$ V. But (Prop. III. Cor. 1.) P $\propto \frac{R}{V}$, therefore FP $\propto \frac{FR}{V}$, therefore $\frac{FR}{V} \propto$ V, and F $\propto \frac{VV}{R}$. Fig. 1.

Cor. 3. *If the centripetal forces are equal; the velocities are as the periodic times; and the radii as the squares of the periodic times, or as the squares of the velocities.*

Cor. 4. *If the centripetal forces be as the radii, the periodic times will be equal.*

For F $\propto \frac{R}{PP}$, and $\frac{F}{R} \propto \frac{1}{PP}$, and if $\frac{F}{R}$ be a given ratio, $\frac{1}{PP}$ will be given, as alſo P.

Cor. 5. *If the centripetal forces be reciprocally as the squares of the diſtances; the squares of the periodical times will be as the cubes of the diſtances; and the velocities reciprocally as the square roots of the diſtances.*

For writing $\frac{1}{RR}$ for F, then $\frac{1}{RR} \propto \frac{R}{PP}$, and $\frac{R^3}{P^2}$ a given quantity. And $\frac{1}{RR} \propto \frac{VV}{R}$, and $\frac{1}{R} \propto$ VV, or $\sqrt{\frac{1}{R}} \propto$ V.

PROP. VI.

If ſeveral bodies revolve in circles, about the ſame or different centers; the radii are directly as the centripetal forces, and the ſquares of the periodic times. 1.

For (Prop. II.) putting the ſame letters as before, P $= \pi t \sqrt{\frac{2R}{s}} = \pi t \sqrt{\frac{2R}{F}}$, and PP $= \pi\pi tt \times \frac{2R}{F}$, and PPF $= 2\pi\pi tt$R $\propto$ R.

Cor.

Fig. 1. Cor. 1. *The radii are directly as the velocities and periodic times.*

For (Prop. IV. Cor. 1.) PF $\propto$ V, but PPF $\propto$ R; therefore PV $\propto$ R.

Cor. 2. *The radii are as the squares of the velocities directly, and the centripetal forces reciprocally.*

For (Prop. III. Cor. 2.) P $\propto \frac{V}{F}$, but (Cor. 1.) R $\propto$ PV; therefore R $\propto \frac{VV}{F}$.

Cor. 3. *If the radii are equal; the centripetal forces are as the squares of the velocities, and reciprocally as the squares of the periodic times. And the velocities reciprocallly as the periodic times.*

For if R be given, $\frac{VV}{F}$, and PPF, and PV, are given quantities, and F $\propto$ VV, or F $\propto \frac{1}{PP}$, and V $\propto \frac{1}{P}$.

SCHOLIUM.

The converse of all these propositions and corollaries are equally true. And what is demonstrated of centripetal forces, is equally true of centrifugal forces, they being equal and contrary.

PROP. VII.

1. *The quantities of matter in all attracting bodies, having others revolving about them in circles; are as the cubes of the distances directly, and the squares of the periodical times reciprocally.*

Let M be the quantity of matter in any central attracting body. Then since it appears, from all astronomical observations, that the squares of the periodical times are as the cubes of the distances, of the planets, and satellites from their respective centers.

centers. Therefore (Cor. 6. Prop. III.) the centripetal forces will be reciprocally as the squares of the distances; that is, $F \propto \frac{1}{RR}$. And (Prop. I.) the attractive force at a given distance, is as the body M, therefore the absolute force of the body M is as $\frac{M}{RR}$. And (Prop. V.) since $F \propto \frac{R}{PP}$, put $\frac{M}{RR}$ instead of F, and we have $\frac{M}{RR} \propto \frac{R}{PP}$, and $M \propto \frac{R^3}{P^2}$. Fig. 1.

Cor. 1. *Hence instead of* F *in any of the foregoing propositions and their corollaries, one may substitute* $\frac{M}{RR}$, *which is the force that the attracting body in* C, *exerts at* A.

Cor. 2. *The attractive force of any body, is as the quantity of matter directly, and the square of the distance reciprocally.*

PROP. VIII.

If the centripetal force be as the distance from the center C. *A body let fall from any point* A, *will fall to the center in the same time, that a body revolving in the circular orbit* ALEA, *at the distance* CA, *would describe the quadrant* AGL. 3.

The truth of this is very readily shewn by fluxions; thus, put AC $= r$, AH $= x$, $t =$ time of describing AH, $v =$ the velocity at H. F $=$ force at H, which is as CH or $r - x$. Then (Mechan. Cor. 2. Prop. V.) the velocity generated is as the force and time; that is, $\dot{v} \propto F\dot{t}$. Also (Mechan. Prop. III. Cor. 1.) the time is as the space divided by the velocity;

Fig. 3. ty; that is, $\dot{t} \propto \frac{\dot{x}}{v}$; therefore $\dot{v} \propto \frac{F\dot{x}}{v} \propto \frac{\overline{r-x} \times \dot{x}}{v}$,

and $v\dot{v} \propto \overline{r-x} \times \dot{x}$, and the fluent is $\frac{vv}{2} \propto rx - \frac{xx}{2}$, or $vv \propto 2rx - xx$, and $v \propto \sqrt{2rx - xx}$ or HG; that is, the velocity at H is as the ordinate HG of the circle.

Now it is evident, that in the time the revolving body describes the infinitely small arch AF, the falling body will descend thro' the versed sine AD, and would describe twice AD in the same time, with the velocity in D. Therefore we shall have,

velocity at F : velocity at D : : AF or FD : 2AD,
and velocity at D : velocity at H : : AF or FD : GH,

therefore,

velocity at F or G : velocity at H : : AF^2 : 2AD $\times$ GH : : $\frac{AF^2}{2AD}$: GH : : CA or CG : GH. But drawing an ordinate infinitely near GH; by the nature of the circle, it will be, as GC : GH : : so the increment of the curve AG : to the increment of the axis AH. And therefore, vel. at G : vel. at H : : as the increment of AG : to the increment of AH. Therefore since the velocities are as the spaces described, the times of description will be equal; and the several parts of the arch AGL are described in the same times as the correspondent parts of the radius AHC. And by composition, the arch AG and abscissa AH, as also the quadrant AL and radius AC, are described in equal times.

Cor. 1. *The velocity of the descending body at any place* H, *is as the sine* GH.

Cor. 2. *And the time of descending thro' any versed sine* AH, *is as the corresponding arch* AG.

Cor.

Cor. 3. *All the times of falling from any altitudes whatever, to the center* C, *will be equal.* Fig. 3.

For these times are $\frac{1}{4}$ the periodic times; and (Prop. V. Cor. 4.) these periodic times are all equal.

Cor. 4. *In the time of one revolution, the falling body will have moved thro'* C *to* E, *and back again thro'* C *to* A, *meeting the revolving body again at* A.

Cor. 5. *The velocity of the falling body at the center* C, *is equal to the velocity of the revolving body.*

For the velocities are as the lines GH and GC; and these are equal, when G comes to L.

PROP. IX.

If a pendulum AB *be suspended at* A, *and be made to revolve by a conical motion, and describe the circle* BEDH *parallel to the horizon.* 4.

Put $\pi = 3.1416$; $p = 16\frac{1}{12}$ *feet, the space descended by gravity in the time* t.

Then the periodical time of B *will be* $\pi t\sqrt{\frac{2AC}{p}}$.

For (Mechan. Prop. VIII.) if the axis AC represents the weight of the body, AB will be the force stretching the string, and BC the force tending to the center C. Also (Mechan. Prop. VI.) if the time is given, the space described will be as the force; whence AC : BC : : p : : $\frac{BC}{AC}p =$ the space descended towards C, by the force BC, in the time t. This is the space s in Prop. II. Therefore instead of s put its value in the periodical time, and (by Prop. II.) we shall have the periodical time of the pendulum $= \pi\ t\sqrt{\frac{2R}{s}} = \pi\ t\sqrt{2BC \times \frac{AC}{BC \times p}}$

$= \pi\ t\sqrt{\frac{2AC}{P}}$.

Cor.

Fig. 4. Cor. 1. *In all pendulums, the periodic times are as the square roots of the hights of the cones,* AC.

For π, t, and p are given quantities.

Cor. 2. *If the hights of the cones be the same, the periodic times will be the same, whatever be the radius of the base* BC.

Cor. 3. *The semiperiodic time of revolution, is equal to the time of oscillation of a pendulum, whose length is* AC, *the hight of the cone.*

For by the laws of falling bodies, $t\sqrt{\frac{AC}{2p}} =$ time of falling thro' $\frac{1}{2}$ AC; and therefore (Mechan. Prop. XXIV.) $1 : \pi : : t\sqrt{\frac{AC}{2p}} : \pi t\sqrt{\frac{AC}{2p}} = \frac{1}{2}\pi t\sqrt{\frac{2AC}{p}}$, the time of vibration, which is half the periodical time.

Cor. 4. *The space descended by a falling body, in the time of one revolution, will be* $\pi\pi \times 2AC$.

For tt (time) : p (hight) : : $\pi\pi tt \times \frac{2AC}{p}$ (per. time) : $\pi\pi \times 2AC =$ hight descended in that time.

Cor. 5. *The periodic time, or time of one revolution, is equal to* $\pi\sqrt{2} \times$ *time of falling thro'* AC.

For the time of falling thro' AC is $t\sqrt{\frac{AC}{p}}$.

Cor. 6. *The weight of the pendulum is to the centrifugal force; as the hight of the cone* AC, *to the radius of the base* CB. *And therefore when the hight* CA *is equal to the radius* CB; *the centripetal or centrifugal force is equal to the gravity.*

PROP.

Fig.

PROP. X.

Suppose a system of bodies A, B, C, *to revolve with a conical motion about the axis* TR *perp. to the horizon, so as to keep the same side always towards the axis of revolution, and the same position among themselves.* 5.

To find the periodical time of the whole system.

1. Let A, B, C be all situated in one plane passing thro' TR. From A, B, C let fall the perpendiculars Aa, Bb, Cc, upon the axis TR. And let A, B, C represent the quantities of matter in the bodies A, B, C. Also put $h = 16\frac{1}{12}$ feet, the hight a body falls in the time t by gravity; $\pi = 3.1416$; P = the periodic time of the system.

By the resolution of forces, Ta (gravity) : Aa (force in direction Aa) : : $h : \frac{Aa}{Ta}\, h$ = space descended by A towards a in the time 1, which is as the velocity generated by the force Aa. Therefore $\frac{Aa}{Ta}\, hA$ = motion generated in A in direction Aa. And the force in direction Aa to move the system towards TR, by the power of the lever TA, is $\frac{Aa}{Ta}\, hA \times Ta$ or $Aa \times hA$. This is the centripetal force of the system, arising from the gravity of A. In like manner the centripetal forces arising from B and C, will be $Bb \times hB$ and $Cc \times hC$.

By the laws of uniform motion, $P : 2\pi \times Aa : : t : \frac{2\pi t \times Aa}{P}$ = arch described by A in the time t. And $\frac{4\pi\pi tt \times Aa^2}{PP \times 2Aa}$ or $\frac{2\pi\pi tt \times Aa}{PP}$ = distance it is drawn from the tangent in that time, or as the velocity generated; and therefore $\frac{2\pi\pi tt \times Aa}{PP} \times A$

=

Fig. 5. = motion of A tending from the center a, by the revolution of the syſtem. And the force in direction aA, to move the syſtem from TR, by the power of the lever TA, will be $\frac{2\pi\pi tt \times Aa}{PP}$ A $\times$ Ta. And this is the centrifugal force of the ſyſtem ariſing from the revolution of A. And in like manner the centrifugal forces ariſing from B and C, will be $\frac{2\pi\pi tt \times Bb}{PP}$ B $\times$ Tb, and $\frac{2\pi\pi tt \times Cc}{PP}$ C $\times$ Tc.

But becauſe the whole ſyſtem always keeps at the ſame diſtance from the axis TF, in its revolution; therefore the ſum of all the centripetal forces muſt be equal to the ſum of all the centrifugal forces. Whence $Aa \times bA = Bb \times bB + Cc \times bC = \frac{2\pi\pi tt}{PP} \times$ into $Aa \times Ta \times A + Bb \times Tb \times B + Cc \times Tc \times C$. And conſequently $P = \pi\ t \sqrt{\frac{2}{b}} \times$

$$\frac{Aa \times Ta \times A + Bb \times Tb \times B + Cc \times Tc \times C}{Aa \times A + Bb \times B + Cc \times C}.$$

2. If the bodies are not all in one plane, let N be the center of gravity of the bodies A, B, C. And thro' N draw the plane TNR; and from all the bodies, let fall perpendiculars upon that plane. Then the periodic time will be the ſame as if all the bodies were placed in theſe points where the perpendiculars cut the plane. For if m be one of the bodies, and mC perp. to the plane. Then the centripetal and centrifugal forces of m in direction cm, will be $cm \times bm$ and $\frac{2\pi\pi tt \times Tc}{PP}\ m \times mc$. But the force cm is divided into the two forces cC, Cm. And all the forces Cm deſtroy one another, becauſe the plane, TNc, paſſes thro' their center of gravity. Therefore the plane is only acted on by the

the remaining force cC. So that the centripetal Fig. 5.
and centrifugal forces will be the same as before,
when the body was placed in C; and the periodic
time is the same.

Cor. 1. *If* Nn *be drawn from the center of gravity perp. to* TF; *then the periodic time of the system,* $P = \pi t \sqrt{\frac{2}{b}} \times$

$$\frac{Ta \times Aa \times A + Tb \times Bb \times B + Tc \times Cc \times C}{Nn \times \overline{A + B + C}}.$$

For (Mechan. Prop. XXXV.) $Aa \times A + Bb \times B + Cc \times C = Nn \times \overline{A + B + C}$.

Cor. 2. *The length of a simple pendulum, making two vibrations, or an exceeding small conical motion, in the same periodic time, will be*

$$\frac{Ta \times Aa \times A + Tb \times Bb \times B + Tc \times Cc \times C}{Nn \times \overline{A + B + C}}.$$

For let TO be the hight of the cone described by the pendulum; then (Prop. IX.) $PP = \frac{2\pi\pi tt}{b} \times TO$; therefore $TO =$

$$\frac{Ta \times Aa \times A + Tb \times Bb \times B + Tc \times Cc \times C}{Nn \times \overline{A + B + C}}.$$

Cor. 3. *If* TO *be the length of an isocronal pendulum, then* O *is the center of gravity of all the peripheries described by* A, B, C; *each multiplied by the body; whether* A, B, C *be the places of the bodies, or the points of projection upon the plane* TNR.

For if $Aa \times A$, $Bb \times B$, $Cc \times C$ be taken for bodies, their center of gravity will be distant from T, the length

$$\frac{Ta \times Aa \times A + Tb \times Bb \times B + Tc \times Cc \times C}{Aa \times A + Bb \times B + Cc \times C}$$ (Mechan.

Fig. 5. chan. Prop. XXXV.) which is equal to TO by Cor. 2. and the peripheries are as the radii, A*a*, B*b*, C*c*.

Cor. 4. *If any of the bodies be on the contrary ſide of the axis* TR, *or above the point of ſuſpenſion* T; *that diſtance muſt be negative.*

Cor. 5. *If any line or plane figure be placed in the plane* TNR; *then the point* O, *which gives the length of the pendulum, will be the center of gravity, of the ſurface or ſolid, deſcribed in its revolution.*

Scholium.

The point O which gives the length of the iſocronal pendulum; is called the *center of turbination* or revolution. And the plane TNR paſſing thro' the center of gravity, the *turbinating plane.*

SECT. II.

The motion of bodies in all ſorts of Curve Lines.

PROP. XI.

THE areas, which a revolving body deſcribes by radii drawn to a fixed center of force, are proportional to the times of deſcription; and are all in the ſame immoveable plane. Fig. 6.

Let S be the center of force; and let the time be divided into very ſmall equal parts. In the firſt part of time let the body deſcribe the line AB; then if nothing hindered, it would deſcribe BK = AB, in the ſecond part of time; and then the area ASB = BSK. But in the point B let the centripetal force act by a ſingle but ſtrong impulſe, and cauſe the body to deſcribe the line BC. Draw KC parallel to SB, and compleat the parallelogram BKC*r*, then the triangle SBC = SBK, being between the ſame parallels; therefore SBC = SBA, and in the ſame plane. Alſo the body moving uniformly, would in another part of time deſcribe C*m* = CB; but at C, at the end of the ſecond part of time, let it be acted on, by another impulſe and carried along the line CD; draw *m*D parallel to CS, and D will be the place of the body after the third part of time; and the triangle SCD = SC*m* = SCB, and all in the ſame plane. After the ſame manner let the force act ſucceſſively at D, E, F, &c. And making D*n* = DC, and E*o* = ED, &c. and compleating the parallelograms as

 before;

Fig 6. before; the triangle $CSm = CSD = DSn = DSE = ESo = ESF$, &c. and all in the ſame immoveable plane. Therefore in equal times equal areas are deſcribed; and by compounding, the ſum of all the areas is as the time of deſcription. Now let the number of triangles be increaſed, and their breadth diminiſhed ad infinitum; and the centripetal force will act continually, and the figure ABCDEF, &c. will become a curve; and the areas will be proportional to the times of deſcription.

Cor. 1. *If a body deſcribes areas proportional to the times, about any point; it is urged towards that point by the centripetal force.*

For a body cannot deſcribe areas proportional to the times, about two different points or centers, in the ſame plane.

Cor. 2. *The velocity of a body revolving in a curve, is reciprocally as the perpendicular to the tangent, in that point of the curve.*

For the area of any of theſe little triangles being given; the baſe (which repreſents the velocity) is reciprocally as the perpendicular.

7. Cor. 3. *The angular velocity at the center of force, is reciprocally as the ſquare of its diſtance from that center.*

For if the ſmall triangles CSD and SBA be equal, they are deſcribed in equal times. The area $CSD = \frac{SC \times CQ}{2}$, and area $SBA = \frac{SB \times BP}{2}$; therefore $SC \times CQ = SB \times BP$. But the angle CSD : angle ASB : : $CQ : cq$: : $SC \times CQ : SC \times cq$: : $SB \times BP : SC \times cq$: : area SBA : area Scq : : $SB^2 : Sc^2$ or SC^2.

PROP.

PROP. XII. Fig. 8.

If a body revolving in any curve VIL, *be urged by a centripetal force tending towards the center* S; *the centripetal force in any point* I *of the curve will be as* $\frac{\dot{p}}{p^3\dot{d}}$; *where* p = *perpendicular* SP *on the tangent at* I, *and* d = *the distance* SI.

For take the point K infinitely near I, and draw the lines SI, SK; and the tangents IP, Kf; and the perpendiculars SP, Sf. Also draw Km, Kn parallel to SP, SI, and KN perp. to SI.

The triangles ISP, IKN. nKm, are similar; as also IKm, IPq. Therefore Iq or IP : IK : : qP : Km. And PS : IP : : Km : mn. And IN : IK : : mn : nK. And multiplying the terms of these three proportions, IP × PS × IN : IK × IP × IK : : qP × Km × mn : Km × mn × nK. That is, PS × IN : IK^2 : : qP : nK $= \frac{Pq \times IK^2}{PS \times IN}$. But (Mechan. Prop. VI.) the space nK, thro' which the body is drawn from the tangent, is as the force and square of the time; that is (Prop. XI.) as the force and square of the area ISK, or as the force × SI^2 × KN^2, or because SI × KN = twice the triangle ISK = IK × SP; therefore nK is as the force × IK^2 × PS^2. Therefore the force at I is as $\frac{nK}{IK^2 \times PS^2} = \frac{Pq \times IK^2}{PS \times IN \times IK^2 \times PS^2} = \frac{Pq}{PS^3 \times IN} = \frac{\dot{p}}{p^3\dot{d}}$.

Cor. 1. *The centripetal force at* I *is as* $\frac{nK}{SI^2 \times KN^2}$, *or as* $\frac{nK}{SP^2 \times IK^2}$.

Fig. 8. Cor. 2. *Hence the radius of curvature in* I, *is* $= \frac{SI \times IN}{Pq}$.

For that radius $= \frac{IK^2}{Km} =$ (by the ſimilar triangles IKm, IqP) $\frac{IK \times IP}{Pq} =$ (by the ſimilar triangles IPS, INK) $\frac{SI \times IN}{Pq}$.

PROP. XIII. *Prob.*

To find the law of the centripetal force, requiſite to make a body move in a given curve line.

Let the diſtance SI $= d$, the perpendicular SP (upon the tangent at I) $= p$; then from the nature of the curve, find the value of p in terms of d, and ſubſtitute it and its fluxion, in the quantity $\frac{\dot{p}}{p^3\dot{d}}$.

Or find the value of $\frac{nK}{SI^2 \times KN^2}$ or $\frac{nK}{SP^2 \times IK^2}$.
Any of theſe will give the law of centripetal force, by the laſt Prop.

Ex. 1.

9. *If a body revolves in the circumference of a circle; to find the force directed to a given point* S.

Draw SI to the body at I, SP perp to the tangent PI, SG perp. to the radius CI. Then SP $=$ GI; becauſe SGIP is a parallelogram. Put SI $= d$, SP $= p$, SC $= a$, CI $= r$, CD $= x$, ID being perp. to SD. Then in the obtuſe angle SCI, $SI^2 = SC^2 + CI^2 + 2SCD$, or $dd = aa + rr + 2ax$; whence $x = \frac{dd - aa - rr}{2a}$. The triangles SCG and CID are ſimilar, whence CI (r) : CD (x)

$(x) : : SC\ (a) : CG = \frac{ax}{r} = \frac{dd - aa - rr}{2r}$; and Fig. 9.

$p = r + \frac{dd - aa - rr}{2r} = \frac{dd + rr - aa}{2r}$; and

$\dot{p} = \frac{d\dot{d}}{r}$. Therefore the force $\left(\frac{\dot{p}}{p^3\dot{d}}\right)$ is as $\frac{d\dot{d}}{rp^3\dot{d}} =$ $\frac{d}{rp^3} = \frac{d \times 8r^3}{r \times \overline{dd + rr - aa}^3}$; that is, the force is as $\frac{d}{\overline{dd + rr - aa}^3}$.

And if $a = r$, the force is as $\frac{1}{d^5}$.

Ex. 2.

If a body revolves in an ellipsis; to find the force tending to the center C. 10.

Let $\frac{1}{2}$ transverse CV $= r$, $\frac{1}{2}$ conjugate CD $= c$, draw CI $= d$, and its semiconjugate CR $= b$. Then by the properties of the ellipsis (Con. Sect. B. I. Prop. XXXIV.) $bb + dd = rr + cc$, whence $b = \sqrt{rr + cc - dd}$; and (ib. Prop. XXXVII.) b or $\sqrt{rr + cc - dd} : c : : r : p =$ $\frac{cr}{\sqrt{rr + cc - dd}}$; and $\dot{p} = \frac{crd\dot{d}}{\overline{rr + cc - dd}|^{\frac{3}{2}}}$. Therefore $\frac{\dot{p}}{p^3\dot{d}} = \frac{crd\dot{d}}{\overline{rr + cc - dd}|^{\frac{3}{2}}} \times \frac{\overline{rr + cc - dd}|^{\frac{3}{2}}}{c^3r^3\dot{d}} = \frac{d}{ccrr}$. Therefore the force is directly as the distance CI.

After the same manner, the force tending to the center of an hyperbola, will be found $\frac{-d}{ccrr}$, which is a centrifugal force, directly as the distance.

Ex. 3.

If a body revolves in an ellipsis, to find the law of centripetal force, tending to the focus S. 11.

Fig. 11. Let the semitransverse OV $= r$, the semiconjugate OD $= c$, draw SI $= d$; and OI, and its conjugate OK $= b$.

Then (Con. Sect. B. I. Prop. XXXV.) $2rd - dd = bb$; and (ib. Prop. XXXVI.) b or $\sqrt{2rd - dd} : c :: d : p = \frac{cd}{\sqrt{2rd - dd}}$; and $\dot{p} =$

$$\frac{c\dot{d}\sqrt{2rd - dd} - cd \times \overline{2rd - dd}^{-\frac{1}{2}} \times \overline{r\dot{d} - d\dot{d}}}{2rd - dd} =$$

$$\frac{c\dot{d} \times \overline{2rd - dd} - cd \times \overline{r\dot{d} - d\dot{d}}}{\overline{2rd - dd}^{\frac{3}{2}}} = \frac{crd\dot{d}}{\overline{2rd - dd}^{\frac{3}{2}}}.$$

Therefore $\frac{\dot{p}}{p^3\dot{d}} = \frac{crd\dot{d} \times \overline{2dr - dd}^{\frac{3}{2}}}{c^3d^3\dot{d} \times \overline{2dr - dd}^{\frac{3}{2}}} = \frac{crd}{c^3d^3} = \frac{r}{ccdd}$.

Therefore the centripetal force is as $\frac{1}{dd}$, or reciprocally as the square of the distance.

Ex. 4.

12. *If a body revolves in the hyperbola* VI; *to find the law of centripetal force, tending to the focus* S.

Draw SI, and the tangent IT, and SP perp. upon it. And let the semitransverse SO $= r$, semiconjugate $= c$, SI $= d$, SP $= p$, and $b =$ semiconjugate to IO.

Then (Con Sect. B. II. Prop. XXXI.) $2rd + dd = bb$, and $b = \sqrt{2rd + dd}$. And (ib. Prop. XXXII.) b or $\sqrt{2rd + dd} : c :: d : p = \frac{cd}{\sqrt{2rd + dd}}$; whence $\dot{p} =$

$$\frac{c\dot{d} \times \sqrt{2rd + dd} - cd \times \overline{2rd + dd}^{-\frac{1}{2}} \times \overline{r\dot{d} + d\dot{d}}}{2rd + dd}$$

$$= \frac{c\dot{d} \times \overline{2rd + dd} - cd \times \overline{r\dot{d} + d\dot{d}}}{\overline{2rd + dd}^{\frac{3}{2}}} = \frac{crd\dot{d}}{\overline{2rd + dd}^{\frac{3}{2}}}.$$

There-

Therefore $\frac{\dot{p}}{p^3\dot{d}} = \frac{crd\dot{d} \times \overline{2rd + dd}|^{\frac{1}{2}}}{\overline{2rd + dd}|^{\frac{1}{2}} \dot{d} \times c^3d^3} = \frac{crd}{c^3d^3} =$ Fig. 12.

$\frac{r}{ccdd}$. Therefore the centripetal force is as $\frac{r}{ccdd}$ or $\frac{1}{dd}$; that is, reciprocally as the square of the distance.

And in like manner the force towards the other focus F, is $\frac{-r}{ccdd}$, or as $\frac{-1}{dd}$, which is a centrifugal force reciprocally as the square of the distance.

Ex. 5.

If a body revolves in the parabola VI; *to find the force tending to the focus* S. 13.

Draw IS, and the tangent IT, and SP perp. to it. And put SI $= d$, SP $= p$, latus rectum $= r$. Then (Con. Sect. B. III. Prop. II. and Cor. 3. Prop. XII.) $pp = \frac{1}{4}rd$, and $2p\dot{p} = \frac{1}{4}r\dot{d}$; and $\dot{p} = \frac{r\dot{d}}{8p} = \frac{r\dot{d}}{4\sqrt{rd}}$. And $\frac{\dot{p}}{p^3\dot{d}} = \frac{r\dot{d}}{4\sqrt{rd} \times \frac{1}{8}rd\sqrt{rd} \times \dot{d}} = \frac{8r}{4rrdd} = \frac{2}{rdd}$. Therefore the centripetal force is reciprocally as the square of the distance CI.

Hence, in all the Conic Sections, the centripetal force tending to the focus, is reciprocally as the square of the distance from the focus.

Ex. 6.

Let VI *be the logarithmic spiral, to find the force tending to the center* S. 14.

Draw the tangent IP, and SP perp. to it, let SI $= d$, SP $= p$; then the ratio of d to p is always

Fig. 14. ways given, ſuppoſe as m to n. Then $p = \frac{n}{m}d$, and $\dot{p} = \frac{n}{m}\dot{d}$. Conſequently $\frac{\dot{p}}{p^3\dot{d}} = \frac{n\dot{d}}{m} \times \frac{m^3}{n^3d^3\dot{d}} = \frac{mm}{nnd^3}$; and the centripetal force is as $\frac{1}{d^3}$, or reciprocally as the cube of the diſtance.

PROP. XIV.

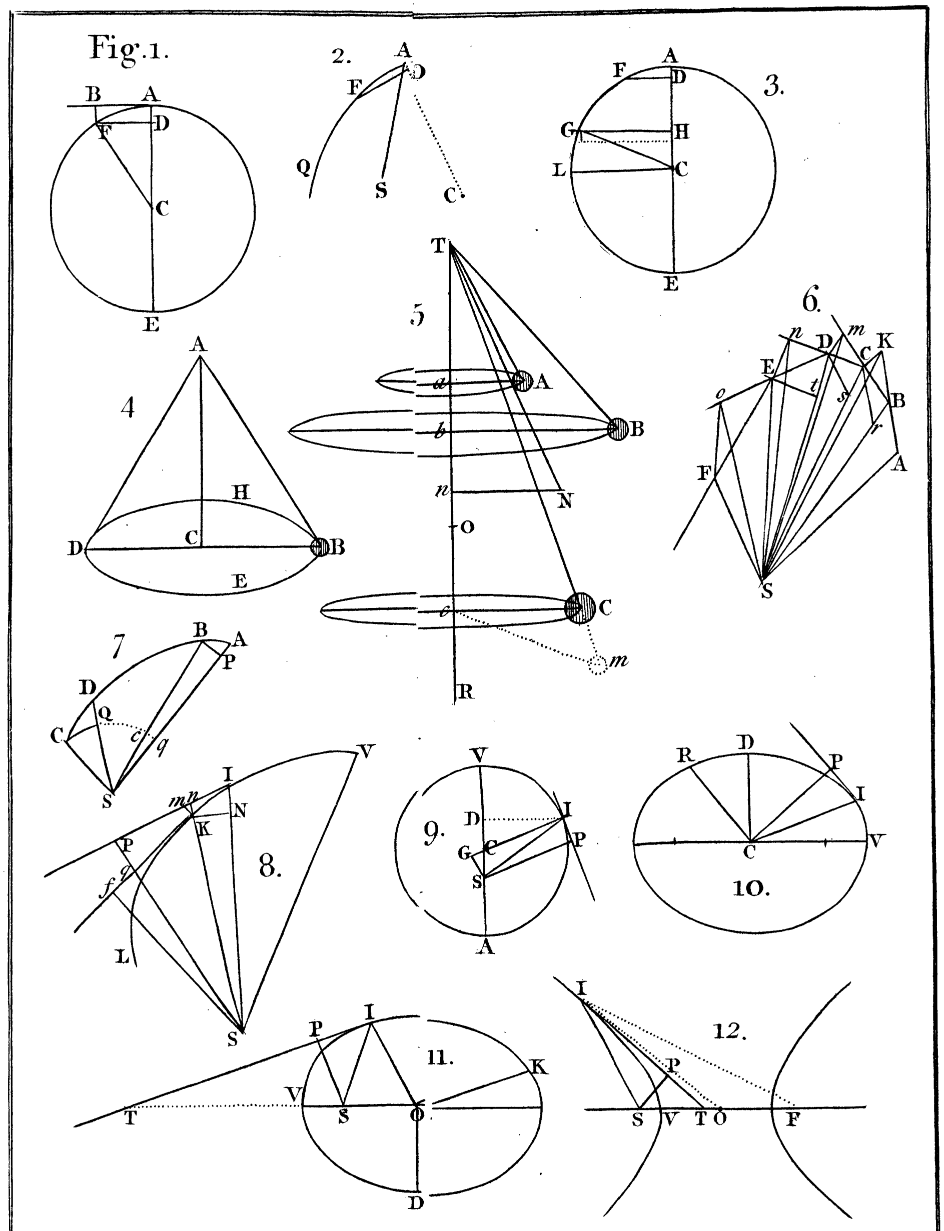
15. *The velocity of a body moving in any curve* QAO, *in any point* A; *is to the velocity of a body moving in a circle at the ſame diſtance; as* $\sqrt{p\dot{d}}$ *to* $\sqrt{d\dot{p}}$. *Putting* $d =$ *diſtance* SA, *and* $p =$ SP *the perpendicular on the tangent at* A.

Let AR be the radius of curvature; from the point a in the curve infinitely near A, draw am, an parallel to AS, AR. Let C = velocity in the curve, c = velocity in the circle. By ſimilar triangle SP (p) : SA (d) : : an : am : : centripetal force tending to R : centripetal force tending to S : : (Prop. V. Cor. 2.) $\frac{CC}{AR} : \frac{cc}{AS}$. But (Prop. XII. Cor. 2.) AR $= \frac{d\dot{d}}{\dot{p}}$; whence $p : d : : \frac{CC\dot{p}}{d\dot{d}} : \frac{cc}{d} : : CC\dot{p} : cc\dot{d}$. And $p\dot{d} : d\dot{p} : : CC : cc$.

Cor. 1. *If* r = *half the tranſverſe axis of an ellipſis; then the velocity of a body revolving round the focus, is to that in a circle at the ſame diſtance; as* $\sqrt{2r - d} : \sqrt{r}$.

For $\dot{p} = \frac{cr\dot{d}}{\overline{2rd - dd}|^{\frac{3}{2}}}$ (See Ex. 3. Prop. XIII.),

and



Cent. Forces. Pl. I. *pa.* 26.

and $p = \frac{cd}{\sqrt{2rd - dd}}$. And the squares of the velocities in the curve, and in the circle, are as Fig. 15.

$\frac{cd\dot{d}}{\sqrt{2rd - dd}}$ and $\frac{crdd\dot{d}}{\overline{2rd - dd}|^{\frac{3}{2}}}$, or as 1 and $\frac{rd}{2rd - dd}$, or as $2r - d$ to r.

Cor. 2. *Suppose as before, the velocity of a body revolving round the center of an ellipsis, is to the velocity in a circle at the same distance; as half the conjugate diameter to that distance, is to the distance.*

For $p = \frac{cr}{\sqrt{rr + cc - dd}}$, and $\dot{p} =$

$\frac{crd\dot{d}}{\overline{rr + cc - dd}|^{\frac{3}{2}}}$. Whence, the squares of these velocities are as $\frac{cr\dot{d}}{\sqrt{rr + cc - dd}}$ and $\frac{crdd\dot{d}}{\overline{rr + cc - dd}|^{\frac{3}{2}}}$

or as 1 to $\frac{dd}{rr + cc - dd}$, or as $rr + cc - dd$ to dd, or as bb to dd. See Ex. 2. Prop. XIII.

Cor. 3. *The velocity in a parabola round the focus, is to the velocity in a circle at the same distance; as* $\sqrt{2}$ *to* 1.

For $p = \frac{1}{2}\sqrt{rd}$, and $\dot{p} = \frac{r\dot{d}}{4\sqrt{rd}}$ (See Ex. 5. Prop. XIII.) Whence the squares of these velocities are as $\frac{1}{2}\dot{d}\sqrt{rd}$ and $\frac{rd\dot{d}}{4\sqrt{rd}}$, or as $\frac{1}{2}rd$ to $\frac{1}{4}rd$; that is as 2 to 1.

Cor. 4. *The velocity of a body in the logarithmic spiral in any point, is the same as the velocity of a body at the same distance in a circle.*

For

Fig. 15. For $p = \frac{n}{m}d$, and $\dot{p} = \frac{n}{m}\dot{d}$, (Ex. 6. Prop. XIII.)

And the ſquares of the velocities are as $\frac{n}{m}d\dot{d}$ and $\frac{n}{m}d\dot{d}$, that is, equal.

PROP. XV. *Prob.*

16. *To find the force which acting in direction of the ordinate* MP, *ſhall cauſe the body to move in that curve.*

Draw mp parallel and infinitely near MP, and Ms parallel to AP. Then the force is as the ſpace mr, thro' which it is drawn from the tangent, in a given time. But ms is the fluxion and mr the ſecond fluxion of the ordinate PM. Therefore making the fluxion of the time conſtant; or which is the ſame thing, making the fluxion of the axis conſtant; find the ſecond fluxion of the ordinate, which will be as the force.

Ex. 1.

Let the curve be an ellipſis whoſe equation is $y = \frac{c}{r}\sqrt{2rx - xx}$. Putting AP $= x$, PM $= y$, $r =$ ſemitranſverſe, $c =$ ſemiconjugate. Then $\dot{y} = \frac{c}{r} \times \frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$, and $\ddot{y} = \frac{c}{r} \times \frac{-\dot{x}\sqrt{2rx - xx}}{2rx - xx}$

$$\frac{-\frac{c}{r} \times \overline{r - x} \times \overline{r\dot{x} - x\dot{x}} \times \overline{2rx - xx}^{-\frac{1}{2}}}{2rx - xx} = \frac{c}{r} \times$$

$$\frac{-2rx + xx - \overline{r - x}^{2}}{\overline{2rx - xx}|^{\frac{3}{2}}} = \frac{c}{r} \times \frac{-rr}{\overline{2rx - xx}|^{\frac{3}{2}}} = \frac{-cr}{y^{3}}.$$

That is, the force is as $\frac{-1}{y^{3}}$, or reciprocally as the

cube

cube of the ordinate. The same is true of the circle, which is one sort of ellipsis. Fig. 16.

Ex. 2.

Let the curve be a parabola, AP $= x$, PM $= y$, and $rx = yy$; then $r\dot{x} = 2y\dot{y}$, and $2\dot{y}\dot{y} + 2y\ddot{y} = 0$; therefore $y\ddot{y} = -\dot{y}\dot{y}$, and $\ddot{y} = -\frac{\dot{y}\dot{y}}{y} = -\frac{rr\dot{x}\dot{x}}{4yy \times y} = \frac{-rr}{4y^3}$, and the force as $\frac{-1}{y^3}$, or reciprocally as the cube of the ordinate.

PROP. XVI.

If the law of centripetal force be reciprocally as the square of the distance. The velocities of bodies revolving in different ellipses about one common center; are directly as the square roots of the parameters, and reciprocally as the perpendiculars to the tangents at these points of their orbits.

Let d, D be the distances in two ellipses; r, c, l, p; and R, C, L, P, the semitransverse, semiconjugate, latus rectum, and perpendicular in the two ellipses. Then the squares of the velocities in two circles whose radii are d, D, (by Prop. IV. Cor. 2.) will be as $d \times$ force in d, and D $\times$ force in D; that is, as $\frac{d}{dd}$ and $\frac{D}{DD}$ or as $\frac{1}{d}$ and $\frac{1}{D}$.

Then (Prop. XIV. Cor. 1.),
velocity in the ellipsis d : vel. in the circle $d :: \sqrt{2r - d} : \sqrt{r}$.
and vel. in the circle d : vel. in the circle D $:: \sqrt{\frac{1}{d}} : \sqrt{\frac{1}{D}}$.

And (Prop. XIV. Cor. 1.)
vel. in the circle D : vel. in the ellipsis D $:: \sqrt{R} : \sqrt{2R - D}$.
Therefore vel. in the ellipsis d : vel. in the ellipsis D ::

Fig.

$D :: \sqrt{\frac{\overline{2r-d} \times R}{d}} : \sqrt{\frac{\overline{2R-D} \times r}{D}} :: \sqrt{\frac{2r-d}{dr}} : \sqrt{\frac{2R-D}{DR}}$.

But (Con. Sect. B. I. Prop. 36.) $p = c\sqrt{\frac{d}{2r-d}}$, and $p\sqrt{\frac{2r-d}{d}} = c$, and $\sqrt{\frac{2r-d}{d}} = \frac{c}{p}$, and $\sqrt{\frac{2r-d}{dr}} = \frac{c}{p\sqrt{r}} = \frac{\sqrt{\frac{1}{2}l}}{p}$ (because $\frac{cc}{r} = \frac{1}{2}l$ by the Conic Sections.) In like manner $\sqrt{\frac{2R-D}{DR}} = \frac{\sqrt{\frac{1}{2}L}}{P}$. Whence, vel. in the ellipsis d : vel. in the ellipsis $D :: \frac{\sqrt{l}}{p} : \frac{\sqrt{L}}{P}$.

Cor. 1. *Hence the velocities in the two ellipsis, are as* $\sqrt{\frac{2r-d}{dr}}$, and $\sqrt{\frac{2R-D}{DR}}$.

Cor. 2. *Also the squares of the areas described in the same time, are as the parameters.*

For the areas are as the arches $\times$ perpendiculars, or as the velocities $\times$ perpendiculars; that is, as $\frac{\sqrt{l}}{p} \times p$ and $\frac{\sqrt{L}}{P} \times P$, or as $\sqrt{l}$ and $\sqrt{L}$.

Cor. 3. *The velocity of a body in different parts of its orbit is reciprocally as the perpendicular upon the tangent at that point; and therefore is as* $\sqrt{\frac{2r-d}{d}}$.

For the parameter is given.

Cor. 4. *The velocity in a conic section at its greatest or least distance, is to the velocity in a circle at the*

the same distance; as the square root of the parameter, to the square root of twice that distance. Fig.

For here $d = D = p = P$, and $L = 2D$. Therefore the velocity in the ellipsis, to the velocity in the circle; as $\frac{\sqrt{l}}{P} : \frac{\sqrt{2D}}{P} :: \sqrt{l} : \sqrt{2D}$.

Cor. 5. *The velocity in an ellipsis at its mean distance, is the same as in a circle at the same distance.*

For if d be the mean distance, then $p = c$. And if D be the radius of the circle, then $L = 2D$, and $P = D$. Whence, vel. in the ellipsis : to the vel. in the circle $:: \frac{\sqrt{l}}{c} : \frac{\sqrt{2D}}{D} ::$ (because $cc = \frac{1}{2}lr$) $\frac{1}{\sqrt{\frac{1}{2}r}} : \frac{1}{\sqrt{\frac{1}{2}D}} :: \sqrt{D} : \sqrt{r}$. But $D = r$, therefore the velocities are equal.

Cor. 6. *Both the real and apparent velocity round the focus* F, *is greatest at* A, *the nearest vertex; and least at* B, *the remote vertex.* 17.

For the real velocity is reciprocally as the perpendicular, which is least at A and greatest at B. And the apparent velocity at F is reciprocally as the square of the distance from F, which distance is least at A, and greatest at B, (Cor. 2. and 3. Prop. XI.)

Cor. 7. *The same things supposed, and* PC, CK *being semiconjugates; the velocity in the curve, is to the velocity towards the focus* F; *as* CK *to* $\sqrt{CK^2 - CD^2}$. 23.

For vel. in the curve : vel. towards F $:: Pp : pn :: FP : NP :: d : \sqrt{dd - pp}$. But $pp = \frac{ccd}{2r - d}$, and $dd - pp = \frac{2rd - dd - cc}{2r - d}d$. Whence vel. in the curve : vel. towards F $:: d :$

$\sqrt{}$

Fig. 23. $\sqrt{\frac{2rd - dd - cc}{2r - d}}\, d :: \sqrt{2rd - dd} : \sqrt{2rd - dd - cc}$ :: (Con. Sect. B. I. Prop. XXXV.) CK : $\sqrt{CK^2 - CD^2}$.

Cor. 8. *The ascending or descending velocity is the greatest when* FP *is half the latus rectum, or when* FP *is perp. to* AB.

For $\sqrt{2rd - dd} : \sqrt{2rd - dd - cc}$: : vel. in the curve $\left(\frac{\sqrt{2r - d}}{d}\right)$: vel. towards F $=$ $\sqrt{\frac{2rd - dd - cc}{dd}}$, and making the square of this velocity a maximum, then $\frac{2rd - dd - cc}{dd} = m$, and $\overline{2r\dot{d} - 2d\dot{d}} \times dd - 2d\dot{d} \times \overline{2rd - dd - cc} = 0$; and $rd - dd - 2rd + dd + cc = 0$, and $- rd + cc = 0$. whence $d = \frac{cc}{r} =$ half the latus rectum.

Cor. 9. *If* FR, *the distance from the focus to the curve be* $= \sqrt{CA \times CD}$; *then* R *is the place where the angular motion about the focus* F, *is equal to the mean motion.*

For the area of a circle whose radius FR is $= \sqrt{CA \times CD}$ is equal to the area of the ellipsis; and if we suppose them both described in equal times; then the small equal parts at R will be described in equal times; and therefore the angular velocities at F will be equal; and both equal to the mean motion. The angular motion in the ellipsis from B to R will be slower; and from R to A swifter, than the mean motion.

PROP.

Fig.

PROP. XVII.

If the centripetal forces be reciprocally as the ſquares of the diſtances; the periodic times in ellipſes, will be in the ſeſquiplicate ratio of the tranſverſe axes AB; *or the ſquares of the periodic times, will be as the cubes of the mean diſtances* FD, *from the common center.* 17.

Put the ſimbols as in the laſt, and t, T, for the periodical times. Then by the nature of the ellipſis $cc = \frac{1}{2}lr$, and $c = \sqrt{\frac{1}{2}lr}$, and $rc = r\sqrt{\frac{1}{2}lr}$. And for the ſame reaſon $RC = R\sqrt{\frac{1}{2}LR}$. Alſo (Prop. XVI. Cor. 2.) the areas deſcribed in the ſame time are as the ſquare roots of the parameters; and therefore the whole areas of the ellipſes, are as the periodical times multiplied by the ſquare roots of the parameters. But the whole areas are alſo as the rectangles of the axes; therefore the rectangles of the axes are as the periodical times multiplied by the ſquare roots of the parameters; that is, rc or $r\sqrt{\frac{1}{2}lr}$: RC or $R\sqrt{\frac{1}{2}LR} :: t\sqrt{l} : T\sqrt{L}$. And ſquaring, $\frac{1}{2}lr^3 : \frac{1}{2}LR^3 :: ttl : TTL$. That is, $r^3 : R^3 :: tt : TT$. And $t : T :: r^{\frac{3}{2}} : R^{\frac{3}{2}} :: \overline{2r}^{\frac{3}{2}} : \overline{2R}^{\frac{3}{2}}$.

Cor. 1. *The areas of the ellipſes are as the periodic times multiplied by the ſquare roots of the parameters.*

Cor. 2. *The periodic time in an ellipſis, is the ſame as in a circle, whoſe diameter is equal to the tranſverſe axis* AB; *or the radius equal to the mean diſtance* FD.

Cor. 3. *The quantities of matter in central attracting bodies, that have others revolving about them in ellipſes; are as the cubes of the mean diſtances, divided by the ſquares of the periodical times.*

 For

Fig. 17. For (Cor. 2.) the periodic times are the ſame when the mean diſtances are equal to the radii; and the reſt follows from Prop. VII.

PROP. XVIII.

18. *If the centripetal forces be directly as the diſtances; the periodic times of bodies moving in ellipſes round the ſame center, will be all equal to one another.*

Let AEL be an ellipſis, AGL a circle on the ſame axis AL, C the center of both. Draw the tangent AD, and *np*F parallel to it, and D*n*, B*p* parallel to AC : AF being very ſmall. Then *Dn* equal to B*p* will be as the centripetal force; and therefore AD and AB, or A*n* and A*p* will be deſcribed in the ſame time, in the circle and ellipſis. Conſequently the areas deſcribed in theſe equal times will be A*n*C and A*p*C. But theſe areas are to one another as *n*F to PF, or as GC to EC; that is, as the area of the circle AGL to the area of the ellipſis AEL. Therefore ſince parts proportional to the wholes are deſcribed in equal times; the wholes will be deſcribed in equal times. And therefore the periodic times, in the circle and ellipſis, are equal.

But (Prop. V. Cor. 4.) the periodic times in all circles are equal, in this law of centripetal force; and therefore the periodic times in all ellipſes are equal.

Cor. *The velocity at any point* I *of an ellipſis, is as the rectangle of the two axes* AC, CE; *divided by the perpendicular* CH, *upon the tangent at* I.

For the arch I × CH is as the area deſcribed in a ſmall given part of time, and that is as the whole area (becauſe the periodic times are equal) or as AC × CE. And therefore the arch I or the velocity, is as $\frac{AC \times CE}{CH}$.

Fig. 18.

PROP. XIX.

The densities of central attracting bodies, are reciprocally as the cubes of the parallaxes of the bodies revolving about them (as seen from these central bodies), and reciprocally as the squares of the periodic times.

For the density multiplied by the cube of the diameter, is as the quantity of matter; that is (by Prop XVII. Cor. 3.) as the cube of the mean distance divided by the square of the periodical time of the revolving body. And therefore the density is as the cube of the distance, divided by the cube of the diameter, and by the square of the periodic time. But the diameter divided by the distance is as the angle of the paralax; therefore the density is as 1 divided by the cube of the paralax, and the square of the periodic time.

PROP. XX.

If two bodies A, B, *revolve about each other*; 19.
they will both of them revolve about their center of gravity.

Let C be the center of gravity of the bodies A, B, acting upon one another by any centripetal forces. And let AZ be the direction of A's motion; draw BM parallel to AZ, for the direction of B. And let AZ, BH be described in a very small part of time, so that AZ may be to BH, as AC to BC; and then C will be the center of gravity of Z and H, because the triangles ACZ and BCH are similar. Whence AC : CB : : ZC : CH. But as the bodies A and B attract one another, the spaces A*a* and B*b* they are drawn thro', will be reciprocally

Fig. 19. as the bodies, or directly as the distances from the center of gravity; that is, $Aa : Bb :: AC : BC$. Compleat the parallelograms Ac and Bd; and the bodies, instead of being at Z and H, will be at c and d. But since $AC : BC :: Aa : Bb$. By division $AC : BC :: aC : bC$. But $AC : BC :: AZ : BH :: ac : bd$. Whence $aC : bC :: ac : bd$. Therefore the triangles cCa, and dCb are similar, whence $Cc : Cd :: ac : bd :: AC : BC :: B : A$. Therefore C is still the center of gravity of the bodies at c and d.

In like manner, producing Bd and Ac, till dg be equal to Bd, and cq to Ac; and if cf, dh, be the spaces drawn thro' by their mutual attractions; and if the parallelograms ce, di, be compleated. Then it will be proved by the same way of reasoning, that C is the center of gravity of the bodies at q and g, and also at e and i, where A describes the diagonals Ac, ce, &c. and B the diagonals Bd, di, &c. and so on ad infinitum.

If one of the bodies B is at rest whilst the other moves along the line AL. Then the center of gravity C will move uniformly along the line CO parallel to AL. Therefore if the space the bodies move in, be supposed to move in direction CO, with the velocity of the center of gravity; then the center of gravity will be at rest in that space, and the body B will move in direction BH parallel to CO or AZ; and then this case comes to the same as the former. Therefore the bodies will always move round the center of gravity, which is either at rest, or moves uniformly in a right line.

If the bodies repel one another; by a like reasoning it may be proved that they will constantly move round their center of gravity.

If the lines CA, Cc, Ce, &c. be equal; and CB, Cd, Ci, &c. also equal. Then it is the case of two bodies joined by a rod or a string; or of one

one body composed of two parts. This body or Fig.
bodies will always move round their common cen- 19.
ter of gravity.

Cor. 1. *The directions of the bodies in opposite points of the orbits, are always parallel to one another.*

For since AZ : Zc : : BH : Hd; and AZ, Zc parallel to BH, Hd; therefore the $\angle ZAc = \angle HBd$, and Bd parallel to Ac. And for the same reason di is parallel to ce, &c.

Cor. 2. *Two bodies, acting upon one another by any forces; describe similar figures about their common center of gravity.*

For the particles Ac, Bd of the curves are parallel to one another, and every where proportional to the distances of the bodies AC, BC.

Cor. 3. *If the forces be directly as the distances; the bodies will describe concentrical ellipses round the center of gravity.*

Cor. 4. *If the forces be reciprocally as the squares of the distances; the bodies will describe similar ellipses or some conic sections, about each other, whose center of gravity is in the focus of both.*

PROP XXI.

If two bodies S, P *attract each other with any* 20.
forces, and at the same time revolve about their center of gravity C. *Then if either body* P, *with the same force, describes a similar curve about the other body* S *at rest; its periodical time, will be to the periodical time of either about the center of gravity; as the square root of the sum of the bodies* ($\sqrt{S+P}$), *to the square root of the fixed or central body* ($\sqrt{S}$).

Let PV be the orbit described about C, and Pv that described about S. Draw the tangent Pr, take

 the

Fig. 20. the arch PQ extremely ſmall, and draw CQR; alſo draw Sqr parallel to CR, and then PQ and Pq will be ſimilar parts of the curves PV and Pv.

Now the times that the bodies are drawn from the tangent thro' the ſpaces QR, qr, with the ſame force, will be as the ſquare roots of the ſpaces QR, qr; that is (becauſe of the ſimilar figures CPRQ and SPrq) as $\sqrt{CP}$ to $\sqrt{SP}$; that is, (by the nature of the center of gravity) as $\sqrt{S}$ to $\sqrt{S+P}$. But the times wherein the bodies are drawn from the tangent thro' RQ, rq, are the times wherein the ſimilar arches PQ, Pq are deſcribed; and theſe times are as the whole periodic times. Therefore the periodic time in PV, is to the periodic time in Pv; as $\sqrt{S}$ to $\sqrt{S+P}$.

Cor. 1. *The velocity in the orbit* PV *about* C, *is to the velocity in the orbit* Pv *about* S; *as* $\sqrt{S}$ *to* $\sqrt{S+P}$.

For the velocities are as the ſpaces divided by the times; therefore, vel. in PV : vel. in Pv : : $\frac{PQ}{\sqrt{S}} : \frac{Pq}{\sqrt{S+P}} :: \frac{CP}{\sqrt{S}} : \frac{SP}{\sqrt{S+P}} :: \frac{S}{\sqrt{S}} : \frac{S+P}{\sqrt{S+P}}$ $:: \sqrt{S} : \sqrt{S+P}$.

Cor. 2. *Bodies revolving round their common center of gravity, deſcribe areas proportional to the times.*

PROP. XXII.

20. *If the forces be reciprocally as the ſquares of the diſtances; and if a body revolves about the center* L *in the ſame periodical time, that the bodies* S, P, *revolve about the center of gravity* C. *Then will* SP : LP : : $\sqrt[3]{S+P} : \sqrt[3]{S}$.

Let PN be the orbit deſcribed about L. Then (Prop. XXI.) per. time in PQ : per. time in Pq : : $\sqrt{S}$:

$\sqrt{S} : \sqrt{S+P} :: \sqrt{CP} : \sqrt{SP}$. And (Prop. XVII.) per. time in Pq : per. time in PN :: $SP^{\frac{3}{2}} : LP^{\frac{3}{2}}$; supposing PQ, PN, similar arches. Therefore per. time in PQ : per. time in PN :: $\sqrt{CP} \times SP^{\frac{3}{2}} : \sqrt{SP} \times LP^{\frac{3}{2}} :: \sqrt{CP \times SP^2} : \sqrt{LP^3}$. But the periodic times are equal; therefore $\sqrt{CP \times SP^2} = \sqrt{LP^3}$, and $LP^3 = CP \times SP^2$, and $LP = \sqrt[3]{CP \times SP^2}$. But $LP : SP :: \sqrt[3]{CP \times SP^2} : SP$ or $\sqrt[3]{SP^3} :: \sqrt[3]{CP} : \sqrt[3]{SP} :: \sqrt[3]{S} : \sqrt[3]{S+P}$. Fig. 20.

Cor. 1. *If the forces be reciprocally as the squares of the distances; the transverse axis of the ellipsis described by* P *about the center of gravity* C, *is to the transverse axis described by* P *about the other body* S *at rest, in the same periodical time; as the cube root of the sum of the bodies* S + P, *to the cube root of the fixed or central body* S.

Cor. 2. *If two bodies attracting each other move about their center of gravity. Their motions will be the same as if they did not attract one another, but were both attracted with the same forces, by another body placed in the center of gravity.*

PROP. XXIII. *Prob.*

Suppose the centripetal force to be directly as the distance. To determine the orbit which a body will describe, that is projected from a given place P, *with a given velocity, in a given direction* PT. 21.

By Ex. 2. Prop. XIII. the body will move in an ellipsis, whose center is C the center of force; and the line of direction PT will be a tangent at the point P. Draw CR perp. to PT. And let the distance CP = d. CR = p, semitransverse axis

Fig. 21. CA = R, semiconjugate axis CB = C. CG (the semiconjugate to CP) = B. f = space a body would descend at P, in a second, by the centripetal force. v = the velocity at P, the body is projected with, or the space it describes in a second. Then $\sqrt{2df}$ = velocity of a body revolving in a circle at the distance CP.

Then (Prop. XIV. Cor. 2.) $v : \sqrt{2df} :: B : d$, and $B\sqrt{2df} = dv$, and $2BBdf = ddvv$, whence $BB = \frac{dvv}{2f}$, and $B = v\sqrt{\frac{d}{2f}}$. But (Con. Sect. B. I. Prop. XXXIV.) $RR + CC = BB + dd = \frac{vvd}{2f} + dd$. And (ib. Prop. XXXVII.) $CR = Bp = pv\sqrt{\frac{d}{2f}}$. Therefore $RR + CC + 2RC = \frac{vvd}{2f} + dd + 2pv\sqrt{\frac{d}{2f}}$, and $R + C =$ $\sqrt{\frac{vvd}{2f} + dd + 2pv\sqrt{\frac{d}{2f}}} = m$. Also $RR + CC - 2RC = \frac{vvd}{2f} + dd - 2pv\sqrt{\frac{d}{2f}}$, and $R - C =$ $\sqrt{\frac{vvd}{2f} + dd - 2pv\sqrt{\frac{d}{2f}}} = n$. Therefore $R = \frac{m+n}{2}$, and $C = \frac{m-n}{2}$.

Then to find the position of the transverse axis AD. Let F, S be the foci. Then (by Con. Sect. B. I. Prop. II. Cor.) we shall have SC or CF = $\sqrt{RR - CC}$. Put FP = x; then SP = $2R - x$, and (ib. Prop. XXXV.) SP × PF or $2Rx - xx = BB$, and $RR - 2Rx + xx = RR - BB$, and $R - x = \pm\sqrt{RR - BB}$; whence $x = R \pm \sqrt{RR - BB}$; that is, the greater part FP = R +

√

$\sqrt{RR - BB}$, and the lesser part SP $= R -$ Fig.
$\sqrt{RR - BB}$. Then in the triangle PCF or PCS, 21.
all the sides are given, to find the angle PCF or PCA.

Cor. *The periodical time in seconds, is* $3.1416 \sqrt{\frac{2d}{f}}$.

For arch $\sqrt{2df}$: time 1″ : : circumference $3.1416 \times 2d : 3.1416 \sqrt{\frac{2d}{f}}$ the periodical time in a circle whose radius is d. And by Prop. XVIII. the periodical time is the same in all circles and ellipses.

PROP. XXIV. *Prob.*

Supposing the centripetal force reciprocally as the 22.
square of the distance; to determine the orbit which a body will describe; that is, projected from a given place P, *with a given velocity, in a given direction* PT.

By Prop. XIII. the body will move in a conic section, whose focus is S the center of force. And the line of direction PT will be a tangent at the point P. Let the distance SP $= d$, transverse axis AD $= z$. $f =$ space a body will descend at P, in a second, by the centripetal force. $v =$ the velocity the body is projected with from P, or the space it describes in a second. Then $\sqrt{2df}$ is the velocity of a body revolving in a circle, at the distance SP.

Then (Prop. XIV. Cor. 1.) $v : \sqrt{2df} : : \sqrt{z - d} : \sqrt{\frac{1}{2}z}$. Whence $v\sqrt{\frac{1}{2}z} = \sqrt{2dfz - 2ddf}$, and $vvz = 4dfz - 4ddf$; and $4dfz - vvz = 4ddf$, whence $z = \frac{4ddf}{4df - vv} =$ AD. And PH $= z - d = \frac{dvv}{4df - vv}$. Therefore if $4df$ is greater than

vv,

Fig. 22. vv, z is affirmative, and the orbit is an ellipsis. But if lesser, z is negative, and the curve is a hyperbola, and if equal, 'tis a parabola.

Draw SR perp. to PT, and let SR $= p$. Also draw from the other focus H, HF perp. to PT. Then (Con. Sect. B. I. Prop. X.) the angle SPR $=$ angle HPF, whence the triangles SPR, HPF are similar; therefore SP (d) : SR (p) : : HP $(z - d)$: HF $= \frac{z-d}{d}p$; and (ib. Prop. XXI.) SR $\times$ HF or $\frac{z-d}{d}pp$ $=$ rectangle DHA or CB^2, the square of half the conjugate axis; therefore CB $=$ $p\sqrt{\frac{z-d}{d}}$.

In the triangle SPH, the angle SPH and the sides SP, PH are given, to find the angle PSH, the position of the transverse axis.

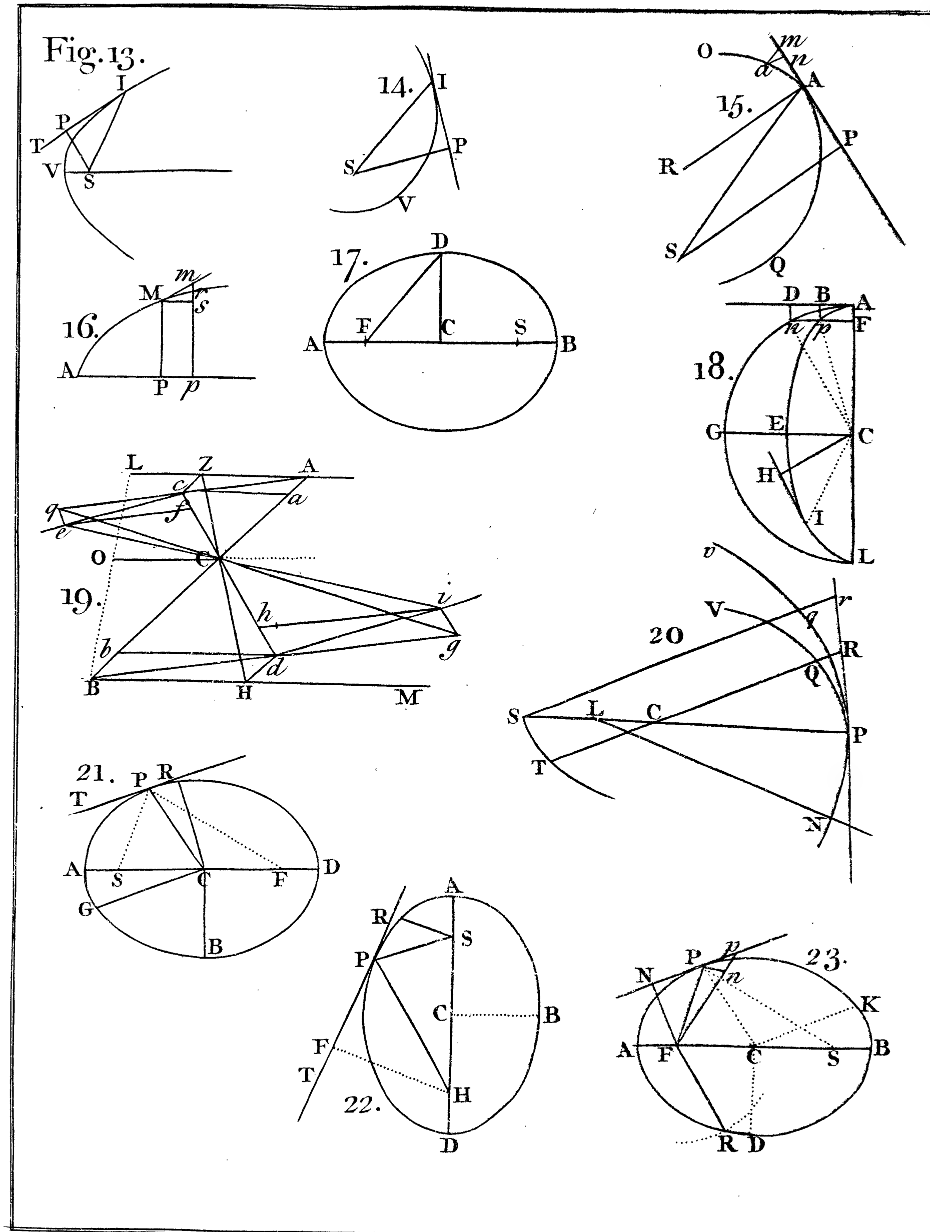
Cor. 1. *The periodical time in the ellipsis* APDB $= 3.1416 \times \frac{4ddf}{\overline{4df - vv}|^{\frac{3}{2}}}$.

For $3.1416\sqrt{\frac{2d}{f}}$ $=$ periodic time in the circle whose radius is d. And (Prop. XVII.) $\overline{2d}|^{\frac{3}{2}} : 3.1416\sqrt{\frac{2d}{f}} : : z^{\frac{3}{2}}$: period. time in the ellipsis $= 3.1416\sqrt{\frac{2d}{f}} \times \overline{\frac{z}{2d}}|^{\frac{3}{2}} = 3.1416 \times \frac{4ddf}{\overline{4df - vv}|^{\frac{3}{2}}}$.

Cor. 2. *The latus rectum of the axis* AD *is* $=$ $\frac{ppvv}{ddf}$.

Cor. 3. *Hence the transverse axis and the periodic time will remain the same, whatever be the angle of direction* SPT.

For



Cent. Forces

Pl. II. pa. 42.

For no quantities but d, f, and v are concerned; all which are given. Fig. 22.

SCHOLIUM.

Some people have dreamed that there may be a ſyſtem of a ſun and planets revolving about it, within any ſmall particle of matter; or a world in miniature. But this cannot be; for though matter is infinitely diviſible; yet the law of attraction of the ſmall particles of matter, not being as the ſquares of the diſtances reciprocally, but nearer the cubes; therefore the revolution of one particle of matter about another, cannot be performed in an ellipſis, but in ſome other curve; where it will continually approach to or recede from the center; and ſo at laſt will loſe its motion. Such motion as theſe can be nothing like that of a ſun and planets.

PROP. XXV.

If a body revolves in the circumference of a circle ZPA, *in a reſiſting medium, whoſe denſity is given. To find the force at any place* P, *tending to the center* C; *as alſo the time, velocity, and reſiſtance. Suppoſing the reſiſtance as the ſquare of the velocity.* 24.

Draw PC, and dp parallel and infinitely near it, cutting the tangent Pd in d. And put CZ $= r$, ZP $= z$, time of deſcribing ZP $= t$, velocity at P $= v$, reſiſtance $=$ R, $f =$ force at P, $g =$ force of gravity at Z, $c =$ velocity in Z. And let a body moving uniformly with the velocity 1, thro' the ſpace 1, in the time 1, meet the reſiſtance 1 in the medium. And let a body deſcend thro' the ſpace a, by the force g at Z, in the ſame time 1.

1. By the laws of uniform motion, the ſpace is as the time × velocity. Whence 1 (ſpace) : 1 × 1 (time × vel.) : : $\dot{z} : v\dot{t} = \dot{z}$, whence $\dot{t} = \frac{\dot{z}}{v}$.

2. By

Fig. 24. 2. By the nature of the circle, $dp = \frac{\dot{z}^2}{2r} = \frac{vv\dot{t}\dot{t}}{2r}$.

3. By accelerated motion, the ſpace is as the force $\times$ ſquare of the time; whence $g \times 1^2$ (force $\times$ time^2) : a (ſpace) : : $f\dot{t}\dot{t}$: dp or $\frac{vv\dot{t}\dot{t}}{2r}$: : $2rf : vv = \frac{2arf}{g}$. And $cc = 2ar$.

4. The velocity generated (or deſtroyed) is as the force $\times$ time; therefore, $g \times 1$ (force $\times$ time) : $2a$ (velocity) : : $R\dot{t} : -\dot{v} = \frac{2aR\dot{t}}{g} = \frac{2aR\dot{z}}{gv}$, and $-v\dot{v} = \frac{2aR\dot{z}}{g}$.

5. The reſiſtance is as the ſquare of the velocity, whence 1^2 (vel.^{2}) : 1 (reſiſtance) : : $vv : R = vv$.

Therefore $-v\dot{v} = \frac{2aR\dot{z}}{g} = \frac{2avv\dot{z}}{g}$. And $-\frac{\dot{v}}{v} = \frac{2a\dot{z}}{g}$, whence $-\log : v = \frac{2az}{g}$, and corrected, $\log : \frac{c}{v} = \frac{2az}{g}$.

Again, ſince $\frac{2arf}{g} = vv$, $\frac{ar\dot{f}}{g} = v\dot{v} = -\frac{2avv\dot{z}}{g}$, and $\dot{f} = -\frac{2vv\dot{z}}{r} = -\frac{4arf\dot{z}}{gr}$, and $-\frac{\dot{f}}{f} = \frac{4a\dot{z}}{g}$, and $-\log : f = \frac{4az}{g}$, and corrected, $\log : \frac{g}{f} = \frac{4az}{g}$.

Alſo $\dot{t} = \frac{\dot{z}}{v} = \frac{-g\dot{v}}{2avv}$, and $t = \frac{g}{2av}$, and corrected $t = \frac{g}{2av} - \frac{g}{2ac}$.

Cor.

Cor. 1. *Hence v = number belonging to the logarithm : log : $c - \frac{2az}{g}$. And f = number belonging to the logarithm : log : $g - \frac{4az}{g}$.* Fig. 24.

Cor. 2. *Therefore the logarithms of v and f, each of them severally decreases equally, in describing equal spaces, ad infinitum. And therefore at every revolution, the log : of v is equally diminished, and likewise that of f. But the body will revolve for ever, for when v is* 0, *t will be infinite.*

Cor. 3. *Hence if the central body at* C, *was so diminished that its log : may decrease equally in describing equal spaces, or in each revolution, after the manner as before-mentioned; then the body will perpetually revolve in a circle, in a medium of uniform density.*

Fig.

SECT. III.

The motion of three bodies acting upon one another; the perturbuting forces of a third body. The motion of bodies round an axis at rest, or having a progressive motion, and other things of the same nature.

PROP. XXVI.

5. *If a body be projected from* A, *in a given direction* AD, *and be attracted to two fixed centers* S, T, *not in the same plane with* AD; *the revolving triangle* SAT, *drawn thro' the moving body, shall describe equal solids in equal times, about the line* ST.

Divide the time into infinitely small equal parts; it is plain that equal right lines AB, BC, CD, &c. would be described in these equal times; and consequently that all the solid pyramids STAB, STBC, STCD, &c. are equal, which would be described in the same equal times; if the moving body was not acted on by the forces S and T.

But let the forces at S and T, act at the end of the several intervals of time; as suppose the force T to act at B in direction BT; so that the body, instead of being at C, is drawn from the line BC, in the direction CF, parallel to BT. And in like manner it is drawn from the line BC, by the force S, in direction CE parallel to SB. And therefore, by the joint forces, the body at the end of the time, must be somewhere in the plane ECF parallel to

SBT,

SBT, as at I. But (Geom. VI. 17.) the solid pyramids STBI and STBC, are equal; being contained between the parallel planes ECF and SBT, and therefore have equal hights; whence STBI = pyramid STAB. Fig. 25.

In like manner continue BI, making IK = BI; and in the next part of time, the body would arrive at K, describing the pyramid STIR equal to STBI. But being drawn from the line IK, by the forces S, T, in the directions KL, KN, parallel to IS, IT; the body will be found at the end of the time, somewhere in the plane LKN parallel to SIT, as suppose at O, and then it will have described the solid STIO = STIK = STBI = pyramid STAB.

And in the same manner producing IO to P, till OP = OI. Then the body, attracted from O, by the forces S, T, will describe another equal pyramid. And so it will continue to describe equal pyramids in equal times; and consequently the whole solids described are proportional to the times of description.

Cor. 1. *When the number of lineolæ* AB, BI, IO, *&c. is increased, and their magnitude diminished, ad infinitum; the orbit* ABIO, *becomes a curve.*

Cor. 2. *Any line* AB *is a tangent at* A, BI *at* B, *&c.* A, B, *&c. being any points in the orbit.*

Cor. 3. *But the orbit* ABIO *is not contained in one plane, except in some particular cases.*

For that the orbit may not deviate from a plane; the forces on both sides thereof, ought to be alike.

PROP.

Fig.

PROP. XXVII.

26. *If the body* T *revolves in the orbit* TH, *about the body* S *at a great distance, whilst a lesser body* P *revolves about* T *very near; and if* C *be the centripetal force of* S *acting upon* T. *Then the disturbing force of* S *upon* P *is* $= \frac{3PK}{ST}C$. *Supposing* PK *parallel, and* KT *perp. to* ST. *And* $\frac{PT}{ST}C$ = *the increase of centripetal force from* P *towards* T.

Let ST $= r$, PT $= a$, PK $= y$, $g =$ force of gravity, $h =$ space descended thereby in time 1. $s =$ the space descended in the time 1, by the force C. $p =$ periodic time of T about S, and $t =$ per. time of P about T. $\gamma =$ centripetal force of T at P, $\pi = 3.1416$.

Since attraction is reciprocally as the square of the distance, then force of S acting at T : force of S acting at P : : $\frac{1}{ST^2} : \frac{1}{SP^2} :: \frac{1}{rr} : \frac{1}{\overline{r-y}^2} :: r : r + 2y$, nearly. And force of S acting at T : to difference of the forces : : $r : 2y$; that is, $r : 2y :: C : \frac{2y}{r}C =$ difference of the forces; and this is the single force by which P is drawn from the orbit QAZ in direction KP or PS.

But since the motion will be the same, whether the single force PS act in the direction PS; or the two forces PT, TS act in the directions PT, TS; substitute these two for that single one; therefore proceeding as before, the force of S acting at T : force of S acting at P : : $\frac{1}{rr} : \frac{1}{\overline{r-y}^2}$. And force

of

of S acting at P in direction PS : force of S acting on P in direction of TS : : PS : TS : : $r - y$: Fig. 26.
$r : : \frac{1}{r} : \frac{1}{r-y}$. Therefore *ex æquo*, force of S acting at T : force of S acting on P in direction TS : : $\frac{1}{r^3} : \frac{1}{\overline{r-y}^3} : : \frac{1}{r^3} : \frac{1}{r^3 - 3r^2y} : : \frac{1}{r} : \frac{1}{r-3y} : : r : r + 3y$, nearly. And the force at T : difference of the forces : : $r : 3y$; or $r : 3y : : C : \frac{3y}{r}C =$ disturbing force of P, acting in direction parallel to TS. And PK (y) : PT (a) : : increase of the disturbing force in direction PK $\left(\frac{y}{r}C\right) : \frac{a}{r}C$, the addition of the centripetal force in direction PT. For when the disturbing force was $\frac{2y}{r}C$, there was no addition of centripetal force at T, but a diminution thereof; as appears by the following Corol.

Cor. 1. *The simple disturbing force, whereby* P *is drawn towards* S, *is* $= \frac{2y}{r}C$. *And the diminution of centripetal force of* P *towards* T, *is* $= \frac{v}{r}C$. *And the accelerating force at* P *in the arch* PA, *is* $= \frac{z}{r}C$. *Putting* $z =$ *sine of* 2PQ, $v =$ *versed sine of* 2PQ.

For let $x =$ PK, and draw KI perp. to PT; then by similar triangles, PT (a) : PK (y) : : PK : PI : : force PK $\left(\frac{2y}{r}C\right)$: force in direction IP or TP $= \frac{2yv}{ar}C = \frac{v}{r}C$.

Also PI (a) : IK (x) : : PK : KI : : force PK

E (2y

Fig. 26. $\left(\frac{2y}{r}C\right)$: force in direction KI or PA $= \frac{2xy}{ar}C = \frac{z}{r}C$. By Trigon. B. I. Prop. II. Schol.

Cor. 2. *The disturbing force at* P *is* $= \frac{q}{59\frac{1}{2}}\gamma$, *q being the sine of the distance from the quadrature,* P *the moon,* S *the sun.*

For (Prop. V.) $C = \frac{ttr}{ppa}\gamma$, and $\frac{3y}{r}C = \frac{3tty}{ppa}\gamma = \frac{3q\gamma}{178\frac{3}{4}}$ (because $\frac{y}{a} = \frac{q}{1}$) $= \frac{q}{59\frac{1}{2}}\gamma$ nearly.

Cor. 3. *If* S *be the sun,* P *a body in the equinoctial of the earth; the disturbing force at* P *is* $= \frac{qg}{12852000}$.

For when P is at the moon's orbit, the force is $\frac{q}{59\frac{1}{2}}\gamma$; but $g = 60 \times 60\gamma$, or $\gamma = \frac{1}{3600}g$, therefore the force becomes $\frac{qg}{59\frac{1}{2} \times 3600}$, and at the earth is $\frac{qg}{59\frac{1}{2} \times 60^3}$.

Cor. 4. *If* S *be the moon,* P *a body on the equinoctial of the earth. The disturbing force at* P *is* $= \frac{qg}{2880000}$.

For the general perturbating force was $\frac{3y}{r}$ C, and here C must be the centripetal force at the moon. Now the centripetal force of the earth, at the distance of the moon is $\frac{1}{60^2}g$. And the moon being 40 times less than the earth, the centripetal force of

of the moon, at the same distance, is $\frac{1}{40 \times 60^2} g$; Fig. 26.
put this for C, then the force of the moon upon the equinoctial, is $\frac{3y}{r} \times \frac{g}{40 \times 60^2} = \frac{3yg}{60a \times 40 \times 60^2}$
$= \frac{qg}{20 \times 40 \times 60^2}$.

Cor. 5. *The disturbing force of the sun, to that of the moon, upon the equinoctial; is as* 1 *to* 4.46.

For these forces are as $\frac{1}{12852000}$ and $\frac{1}{2880000}$ or as 288 to 1285, or as 1 to 4.46.

Cor. 6. *If f be the apparent diameter, and d the density of the perturbating body. Then the disturbing force will always be as df^3y.*

For that force is $\frac{3y}{r}C$ or as $\frac{yC}{r}$. Let its diameter, $= b$, M $=$ its quantity of matter. Then C is as $\frac{M}{rr}$; that is, as $\frac{db^3}{rr}$. Therefore the disturbing force is as $\frac{db^3y}{r^3}$, or as $dy \times f^3$.

Cor. 7. *If* P *be a point in the equator of the earth,* S *the sun.*
The centrifugal force of P :
is to the perturbating force PT : :
As the square of the earth's periodical time about the sun pp :
to the square of the earth's periodical time about its axis tt.

Let $t =$ time of revolution of the earth round its axis; then $t : 2\pi a$ (circumference) : : $1''$: $\frac{2\pi a}{t} =$ arch described in one second; and the versed

Fig. 26. ed ſine $= \frac{4\pi\pi aa}{tt \times 2a} = \frac{2\pi\pi a}{tt} =$ aſcent or deſcent by the earth's centrifugal force. But forces are as their effects, whence $h : g :: \frac{2\pi\pi a}{tt}$ (aſcent) $: \frac{2\pi\pi ag}{tth}$ the centrifugal force itſelf. But the perturbating force is $\frac{a}{r}C = \frac{asg}{rh}$. Whence the centrif. force : perturbating force $:: \frac{2\pi\pi ag}{tth} : \frac{asg}{rh} :: \frac{2\pi\pi}{tt} : \frac{s}{r} ::$ $2\pi\pi r : tts :: \frac{2\pi\pi r}{s} : tt$. But $\frac{2\pi\pi r}{s} = pp$. For $\sqrt{2rs} : 1'' :: 2\pi r : p = \frac{2\pi r}{\sqrt{2rs}}$, and $pp = \frac{4\pi\pi rr}{2rs} = \frac{2\pi\pi r}{s}$.

Cor. 8. *Hence the body* P *is accelerated from the quadratures* Q, Z, *to the ſiziges* A, B; *and retarded from the ſiziges to the quadratures. And moves faſter, and deſcribes a greater area, in the ſiziges than in the quadratures.*

PROP. XXVIII.

The ſame things ſuppoſed as in the laſt Prop. the linear error generated in P *in any time, is as the diſturbing force and ſquare of the time. And the angular error, ſeen from* T, *will be as the force and ſquare of the time directly, and the diſtance* TP *reciprocally.*

For the motion generated in a given part of time, by any force, will be as that force; and in any other time as the force and the ſquare of the time. The motion ſo generated is the linear error of P, as it is carried out of its proper orbit, by the force $\frac{3y}{r}$C. And that error, as ſeen from T, is as the angle

angle it is seen under; and therefore is as that linear Fig.
error, divided by the distance TP; and therefore 26.
is as the force and square of the time, divided by the distance.

Cor. 1. *The linear error generated in one revolution of* P, *is as the disturbing force and square of the periodical time,* $\frac{3ca}{r}tt$. *And the angular error in one revolution is as the force and square of the periodic time divided by the distance.*

Cor. 2. *The mean linear error of* P *in any given time, will be as the force and periodical time,* $\frac{a}{r}Ct$. *And the mean angular one, as the force and periodical time, divided by the distance.*

For let the given time be 1; then t (time) : $\frac{aC}{r}tt$ (whole error) : : 1 : $\frac{aCt}{r}$, the error in the given time.

Cor. 3. *The mean lineal error in any given time, is as* TP *and the periodical time of* P *directly, and the square of the periodical time of* T *reciprocally. And the mean angular error, as the periodical time of* P *directly, and the square of the periodical time of* T *reciprocally.*

For (Prop. V.) C is as $\frac{r}{pp}$, and $\frac{at}{r}$ C is as $\frac{at}{r} \times \frac{r}{pp}$ or $\frac{at}{pp}$. And the angular error as $\frac{t}{pp}$.

Cor. 4. *In any given time, the lineal error is as* TP *and the periodical time of* P *directly, and the cube of* ST *reciprocally. And the angular error as the periodical time of* P *directly, and the cube of* ST *reciprocally.*

Fig. 26. For (Prop. XVII.) pp is as r^3, therefore $\frac{at}{pp}$ is as $\frac{at}{r^3}$.

Cor. 5. *The linear error in a given time is as* $\frac{a^{\frac{5}{2}}}{r}$C, *and the angular error as* $\frac{a^{\frac{3}{2}}}{r}$C.

For t is as a^3, and $\frac{at}{r}$C *as* $\frac{a^{\frac{5}{2}}}{r}$ C.

Cor. 6. *And univerſally, the angular errors in the whole revolution of any ſatellites; are as the ſquares of the periodic times of the ſatellites directly, and the ſquares of the periodic times of their primary planets reciprocally. And the mean angular errors are as the periodical times of the ſatellites, divided by the ſquares of the periodic times of their primary planets.*

For by Cor. 1. the angular error is as the force and ſquare of the time divided by the diſtance; that is, as $\frac{Ca}{r} \times \frac{tt}{a}$; that is, (becauſe C is as $\frac{r}{pp}$) as $\frac{tt}{pp}$. The reſt is proved in Cor. 3.

PROP. XXIX.

27. *If a ſpheroid* AB *revolves about an axis* ST *in free ſpace, which axis is in an oblique ſituation to the ſpheroid; the ſpheroid will, by the centrifugal force, be moved by degrees into a right poſition ab; and afterwards by its libration, into the oblique poſition* $\alpha\beta$. *And then will return back into the poſitions ab,* AB; *and ſo vibrate for ever.*

Let C be the center of the ſpheroid; D the center of gravity of the end ICLB; E that of the end ICLA; Dd, Ee perp. to ST. Then the centrifugal force of the end CB, ſuppoſing it to act wholly

wholly at D, in direction *d*D, having nothing to oppose it, will move the end CB from B towards *b*, with a force which is as C *d*. And at the same time, the centrifugal force of the end CA, acting in direction *e*E, will move the end CA from A towards *a*, conspiring with the motion of the end CB; by which means it will by degrees come into the position *ab*. And then by the motion acquired, it will come into the position $\alpha\,\beta$, making the angle SC α = SCB. And the motion being then destroyed, it will return back, by the like centrifugal forces, acting the contrary way; and be brought again into the positions *ab*, and AB; and continue to vibrate thus perpetually. Fig. 27.

Cor. *Hence if the axis of the earth is not precisely the same as the axis of its diurnal rotation; the earth will have such a libration as is here described, but exceeding small. This is supposing it a solid body; but if it was a fluid, it would by the centrifugal force, form itself into an oblate spheroid.*

PROP. XXX. *Prob.*

If a globe APBQ *in free space, revolve about the axis* SCT, *in direction* ADB; *and if any force applied at* V, *the end of the radius* CV, *acts by a single impulse in direction* VG *perp. to* CV, *in the plane* VCD. *To find the axis about which the globe shall afterwards revolve.* 28.

Suppose the great circle VBQA perp. to the line of direction VG; and if VH, VI be 90 degrees; it is plain, if the first motion was to cease; the globe by the impulse at V, would revolve round the axis IH, which by the first motion was round the axis PQ. Therefore by both motions together it will move round neither of them. Now

Fig since a point of the surface moving with the great-
28. est velocity about ST, will move along the great circle ADB; and a point having the greatest velocity about IH, moves along the great circle VDE. Therefore a point that will have the greatest velocity, by the compound motion, will also be in a great circle passing thro' D. Therefore in the great circles DB, DE, take two very small lines D*r*, D*o*, in the same ratio as the velocities in AD, and VD; and compleat the parallelogram D*opr*. Then thro' D and *p* draw a great circle KD*p*L; and a point having the greatest motion, arising from a composition of the other two motions, will move along KD*p*L. Therefore finding F, R the poles of the circle KD*p*L, FR will be the new axis of revolution, or the axis sought. And the velocity about the axis FR will be proportional to D*p*; VBQA being always supposed perp. to GV, or to the plane DVC.

Note, if you suppose an equal force applied at E, in direction contrary to GV, it will by that means keep the center C of the globe unmoved, and will likewise generate twice the motion in the globe.

Cor. 1. *The greater the force is that is applied to* V, *the greater the distance* PF *is, to which the pole is removed. And if several impulses be made successively at* V, *when* V *is in the circle* APB, *the pole* F *will be moved further and further towards* H, *in the circle* APB.

For several small forces or impulses have the same effect as a single one equal to them all.

29. Cor. 2. *If the force act at* P, *in direction perp. to the plane* CPB; *and* Dr, Do *be as the velocities along* DB *and* DQ. *The great circle* KDL *(passing thro'* D *and p) is the path of the point* D; *and its pole* F, *or axis of revolution* RF; *the pole being translated*

from

from P *to* F. *And if the impulse be exceeding small,* PF *will also be exceeding small.* Fig. 29.

Cor. 3. *If the force at* P *always acts in parallel directions, whilst the globe turns round. The pole will make a revolution in a small circle upon the surface of the globe, in the time of the globe's rotation, and the contrary way to the globe's motion.*

For let a single impulse at P translate the pole to F; and afterwards when the globe has made half a revolution, and the point P is come to *p*; then if a new impulse be made at F, the pole will be translated to *p* which is now P; that is, it will be moved back to its first place on the globe. So that in any two opposite points of rotation, the place of the pole is moved contrary ways, and so is carried back again the same distance. And since the globe revolves uniformly, if the force act uniformly, it will move the pole all manner of ways, or in all manner of directions upon its surface; that is, it will describe a circle, which will end where it begun. And in describing this polar circle, the motion will be contrary to the motion of the globe; for suppose PFB an immovable plane. If the globe stood still, the pole would move in a great circle, in the plane PFB. But since all the points of the globe which come successively to the plane PB, are not yet arrived at it, but are so much further short of it, as PF is greater; 'tis evident all these points will lie on this side the plane PB. And as any fixed point will describe a circle on the moving globe, contrary to the motion of the globe; so will a point that is not fixed, but moving in the plane PFB, likewise describe a circle (or some curve) contrary to the motion of the globe. Or shorter thus, suppose the globe to stand still, and the direction of the force to move backward, then the relative motion will be the same as before; and then

Fig. 29. then the pole F will move backward too, as it will follow the force, being at right angles to it.

Cor. 4. *Since the pole by one impulse is translated to* F; *the new pole* F *is therefore another point of the material globe, distinct from* P. *And the particle* P *that was before at rest, will now revolve about the particle* F *at rest.*

For the new pole F is that particle of the globe which happened to be revolving in F, when the impulse was made at P. The matter of the great circle ADB does not come into the circle KDL, but only the point D of it. For when the force is impressed, the other particles M, N, by the compound motion, will be made to revolve in the directions M*m*, N*n*, parallel to D*p*; and therefore will describe lesser circles about F; whilst only D describes the great circle KDL.

Cor. 5. *What is demonstrated of a sphere is true also of an oblate spheroid, whose axis is* PQ; *and the force impressed at* P, *acting in direction perp. to* CPB, *or parallel to* CD *the radius.*

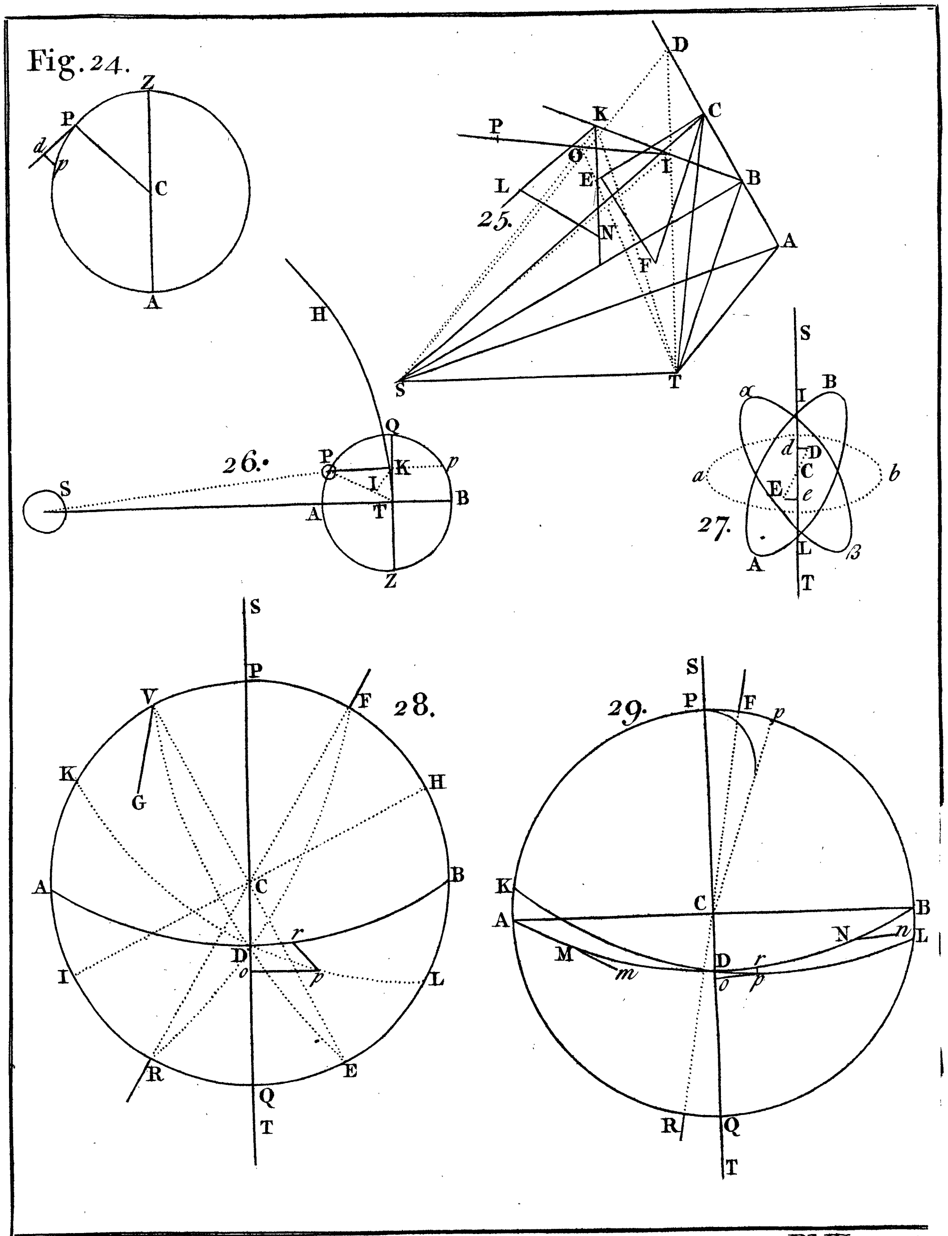
Cor. 6. *But if the force at* P *act in direction contrary to the foregoing (as in case of an oblong spheroid); the pole of rotation will be moved from* P *towards* A, *contrary to the way of the other motion.*

Cor. 7. *And in general the pole* P *will always be moved in a direction perp. to that of the power; and towards the same way as the spheroid revolves.*

Cor. 8. *Hence after every half rotation of the globe round its axis, the places upon the globe change their latitude a little; which, after an entire rotation return to the same quantity. But this variation is so trifling, as to come under no observation.*

This is evident, because the pole is altered; and

of



Cent. Forces

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of consequence, the distance therefrom is altered Fig.
in all places, except in the great circle FCR. 29.

PROP. XXXI. *Prob.*

Let AB *be an oblate spheroid, whose axis is* PC; 30.
and let it revolve round that axis, in the order ADB, *which is its equinoctial; and if any force act at* P, *in direction* PG *perp. to* PC, *and in the plane* P⊙C, *which moves slowly about, in the order* ADB. *To find the motion generated in the spheroid.*

Let EL be an immovable plane like the ecliptic, in which the center C of the spheroid always remains. ON another plane parallel to it. Erect CM perp. to these planes; and make the angle MCN = the angle B♎C, in which the equinoctial B♎ cuts the ecliptic CL. Suppose the spherical surface OVN to be drawn, whose pole is M; and produce CP to cut it at R, in the circle ORN. Then PCD and RCM are in one plane, and both of them perp. to P♎ and P♈C. Now to find the motion of the axis of the spheroid. Here OVN is the upper side of the surface.

This Prop. differs from the last in several respects. The last Prop. regards only the motion of the pole upon the surface of the globe, and that is caused by a motion which is generated in the globe itself. But in this Prop. we consider the motion of the axis CR in the fixed spherical surface OVN; which always proceeds in one direction, as long as the moving force keeps its position. In the last Prop. the motion of the globe round its axis is performed in a very short time; but here the revolution of the force, in the moving plane P⊙C, is a long time in its period.

Now by the last Prop. Cor. 7. the force PG will always move the axis of rotation CPR in a direction

Fig. 30. tion perp. to PG. Therefore ſuppoſe the plane P⊙C to revolve ſlowly round PC, and we ſhall find that in the beginning of the motion, when ⊙ or F is at ♎, then the plane (P⊙C or) P♎C will be perp to the plane CRM, and at that time the motion of R will be directed from R towards M. And when that plane comes to the poſition PF⊙C, the motion of R will be directed to ſome place between N and V. And when it is got to the tropic D, then R's motion is directed along the circumference RV; for then P⊙C coincides with CRM. But when P⊙C arrives at *f*, the motion will be directed from R to ſome point *t* without the circle. And laſtly, when P⊙C is at the other interſection ♈, beyond B; the motion of R will be directed to *m* oppoſite to M. The reſult of all which is, that the pole R will deſcribe ſuch a curve as R1234; and then the ſame force begins again at ♎; which being repeated, another ſimilar figure 456 is deſcribed by R; and ſo on for more. The ſame force I ſay is repeated, for when the plane P⊙C comes on the other ſide of the globe, the force acts the contrary way, and therefore 'tis all one as if it acted on the firſt ſide of the globe.

It muſt be obſerved, that as R moves thro' 1234, the interſection ♎ gradually moves towards E. And as to the force PG, it may be ſuppoſed variable, at different poſitions of the plane P⊙C. And according to the quantity of force in the ſeveral places, different curves (1234) will be deſcribed.

Cor. 1. *Hence it is evident, that the inclination of the planes* ADB *and* ECL, *is greateſt when* P⊙C *paſſes thro'* ♎ *and* ♈. *And leaſt when it paſſes thro' the tropic* D. *And that the inclination decreaſes from the node* ♎ *to the tropic* D, *and increaſes from the tropic to the node.*

For

For when P⊙C is in ♎, R is fartheſt from M; but when it is in D, R is at 2, and its direction is parallel to R4, and then 2M is leaſt. Fig. 30.

Cor. 2. *After a revolution of the plane* PF⊙C *(in which the force always acts), the inclination comes to the very ſame it was at firſt.*

For at any two points F, *f*, equidiſtant on each ſide from the tropic; the force is directed contrary ways, from and to the circle RV; and therefore the motion, in the curve R1234, being alſo equal and contrary, from and towards RV, they mutually deſtroy one another; and therefore after a revolution, or rather half a revolution, the pole R is brought back to the circle RV, and then the angle RCM is the ſame it was at firſt.

Cor. 3. *The motion of the pole* R, *reckoned in the circle* OVN, *is always from* R *towards* V, *then thro'* N, O, *and* R.

For tho' the motion of R towards and from M, in the line M*m*, in one revolution, is equal both ways; and ſo R is always brought to the circle again; yet the motion conſidered along the circle is always in the order RVN. Thus it goes thro' the curve 123 to 4, ſo that after half a revolution of P⊙C, it is advanced forward in the circle RV, the length R4.

Cor. 4. *The motion along the circle is ſometimes faſter and ſometimes ſlower. At* 2 *it moves faſteſt of all; at* R *and* 4, *it moves ſloweſt, or rather is ſtationary for a moment.*

Cor. 5. *The pole* R, *and the nodes move the contrary way about, to what* AB *revolves.*

Cor. 6. *If the force* PG *ſtands ſtill, the pole* R *will ſtill move backwards as before; and that in a right*

Fig. 30. *right line, or rather, a great circle. And if* PG *moves backward thro'* BDFA; *the pole* R *will still go backward; but then the curve* R24, *will be concave towards* M, *like* 87R, *being contrary to the other where the force moved forward.*

Cor. 7. *But in an oblong spheroid, where the force acts in direction* GP, *quite contrary to the other;* R *will describe the curve* R78, *without the circle* ORV; *every particle of it in a contrary direction to these of* R24. *And therefore the pole* R, *and the nodes* ♈ *and* ♎ *will move the same way about as* ADB *revolves, and contrary to what they do in an oblate spheroid.*

For the force being directed the contrary way; of consequence the motion must be so too.

Cor. 8. *And in an oblong spheroid, if the force* GP *move the contrary way about; yet the pole* R *will still move forward. And the curve described by* R, *will have its convexity the contrary way.*

Cor. 9. *Hence if the quantities and proportions of these forces, in different places be known; it will not be difficult to delineate the curve* R1234, *upon the spherical surface* OVNM.

PROP. XXXII. *Prob.*

31. *If a planet (or the moon) move in the orbit* ATE*t*, *round an immovable center* C, *whose plane is inclined to the plane of the ecliptic* AQE; *and a force acts upon it in lines perp. to* GZ, *and parallel to the ecliptic, directed always from the plane* GZ *to either side. To find the motion of the nodes* A, E; *and the variation of the orbit's inclination* PAO.

Let ATZE be half the orbit raised above the ecliptic AQ, AE the line of the nodes; T, *t*, the

the tropics. Draw CM perp, to the ecliptic AQE, and CR perp. to the orbit ATE. Round M as a pole describe a spherical surface RVNX; then RC will be the axis of the orbit, and MC of the ecliptic, or circle AQE. Thro' the points T, C, M, draw the plane RMC; and thro' A, C, M, another plane cutting the circle RNF in X and H; then RF is perp. XH, and the circle RNX parallel to GEQ. Note, RNX is the upper side of the surface. Fig. 31.

Let the planet be at P, and let P2 be the space it describes in any small time, and the line P1 the space it would be drawn thro' in the same time, by the force acting from the plane MGZ. Compleat the parallelogram P123, and P3 will be its direction by the compound force. Now as the line P1 is parallel to the ecliptic, 'tis plain the point 3 will be below the plane of the orbit; and the plane CP2 will be moved into the position CP3, revolving about CP; consequently the axis RC will be moved in a direction perp. to CP. And the pole R will be moved to some point between F and H. This being duly attended to, the motion of the pole R will be known for all the places of P in the orbit GATZ. For about G the motion of R is directed perp. to M*n*; at A it moves perp. to MX, or in direction RM. At T it moves parallel to MH, or in the curve RV. Approaching to Z, it moves perp. from M*n*. So that in the passage of the planet P, from G to Z, thro' GAPTZ; the pole R of its orbit, moves thro' the curve R1234. But in the other half of the orbit ZE*t*G, as the force is directed the contrary way from the plane GZMN, the pole R will return back at 4, and describe a similar curve 45678. So that when the planet P has made one revolution, the pole of its orbit R will be found at 8. But in this position of the nodes, the point 8 will be within the spherical

Fig. rical surface RHNX, not reaching the periphery
31. RV. For the several parts of the curve being described in all directions in respect of the line nN; the points R, 4, will be equidistant from nN, and likewise 4, 8, for the same reason.

Now suppose the force and the plane GZNn to to revolve about the axis CM in the order GATE. Then after it is come to such a position, that the ascending node A is as far on the other side of G, suppose at a; then the pole R will be as far on the other side of V, suppose at r; and being also as far from Nn, on the same side; the curves (12468) will approach VH there, by the same degrees as they receded from it at RV. And therefore the pole R will by degrees be brought to the circle again. Thus in every two correspondent points on each side V, the forces and their effects balance one another, and R will be at the same distance from the circle RVH. And therefore after half a revolution of the plane GZ to the nodes, the angle RCM, and consequently the inclination of the orbit, comes to the same as at first. And likewise as the pole R moves forward or backward in the circle RVH; the motion of the nodes A, E, will be forward or backward.

Cor. 1. *In this position of the nodes at* A *and* E, *the inclination of the orbit* ATE *will be diminished every revolution of* P. *But on the opposite side at* a, *the inclination increases every revolution of* P.

For the points 4, 8, come nearer and nearer to M, and the contrary at r.

Cor. 2. *When the nodes are at* A, E; *the inclination decreases; when the planet is in* GT *or* Zt; *and increases in* TZ *and* tG.

For R moves to 3, whilst P moves thro' GT. At 3 it is at its nearest distance to M; from 3 to

4 R

4 R recedes from M, whilst P moves thro' TZ. Fig.
And the like on the other half of the orbit. 31.

Cor. 3. *When the planet is in* GA *and* ZE, *the nodes go forward. But in* AZ *and* EG, *they go backward.*

For whilst R passes thro' R1, its motion is forward, viz. from R toward *n*, and at 1 where it moves parallel to RM, it is stationary; that is, when P is in A. Thro' 1234, R moves backward, or towards V; and then P is in AZ.

Cor. 4. *In general, the nodes are always regressive, except when* P *is between a node, and its nearest quadrature; and then they are progressive, wherever the nodes are situated.*

Cor. 5. *The nodes go fastest back when the planet is in* T *and t.*

For then R is at 3 and 7.

Cor. 6. *The inclination varies most, when* P *is at* A *and* E.

For then R is at 1 and 5.

Cor. 7. *And from the various situation of the nodes, and the place of* P, *it may easily be determined, when the inclination increases or decreases, in any case.*

Cor. 8. *Hence if the quantities of these forces were known, it would not be difficult to delineate the motion of the pole* R, *upon the spherical surface* RXFH; *and at any time to find the inclination, and place of the node.*

Cor. 9. *And to find the nature of the curve* R1234 *described by the pole* R. *Supposing the force directed always to the sun; and to be as the distance of* P *from the plane* GZ.

Let RDB be the curve, and let the tangent *t*DT 32.
revolve about the curve RDB, beginning at R, so

Fig. as the end T may move uniformly thro' all the
32. points of the compaſs, in the ſame manner as P moves thro' its orbit ATZEt. It is plain this is one property of the curve RDT.

Now ſince the ſun's rays fall at the ſame obliquity upon all parts of the plane ATQ, therefore the force to draw P in a direction parallel to theſe rays, being the ſame at equal diſtances from the plane GZ, and always as the diſtance; therefore by the reſolution of motion, the diſtance that P is drawn perpendicularly from the plane of its orbit, will alſo be as that diſtance; and that is as the variation of the orbit's inclination. Therefore if P, inſtead of moving to 2 move to 3, then the force at P or PB (fig. 31.) will be as the angle 2P3: ſuppoſ-
33. ing the ſun's diſtance from the node to remain the ſame, during one revolution of P.

But when the ſun or the force alters its poſition,
34. it will be greater or leſs on that account, in proportion to the ſine of OL (where OL is perp. to AL), and that is as the ſine of AO, the diſtance from the node, the angle A being given. From hence it follows that univerſally, the force acting on P will be always as BP × S.AO; that is, as S.GP × S.AO (fig. 31.); that is, as the ſine of the diſtance of P from the quadratures, and the ſine of the diſtance of the ſun at O from the node.

Now let us find the nature of the curve R1234,
32. ſuppoſing AO to remain the ſame for one revolution of P. Put RA $= x$, AD $= y$, RD $= z$. Since by the generation of the curve, the angle ADt = arch GP, and the force is as the ſine of GP or of ADt, and $\frac{\dot{x}}{\dot{z}} =$ S.ADt. Alſo it is plain, the increment of the curve at D is as that force; therefore $\frac{\dot{x}}{\dot{z}}$ is as $\dot{z}$. And ſince in paſſing thro' the

parti-

particle of the curve $\dot{z}$, the line D*t* is supposed to change its direction uniformly, therefore the angle of contact is given; whence $\dot{z}$ or $\frac{\dot{x}}{\dot{z}}$ is as the radius of curvature, or as $\frac{\dot{z}\dot{y}}{\ddot{x}}$; that is, $\frac{\dot{x}}{\dot{z}}$ is as $\frac{\dot{z}\dot{y}}{\ddot{x}}$, or $ax\ddot{x} = \dot{z}^2\dot{y}$ ($\dot{z}$ being given), and the fluent is $\frac{a\dot{x}^2}{2} = y\dot{z}^2$, and $\dot{x} : \dot{z} :: \sqrt{y} : \sqrt{\frac{1}{2}a}$ :: as the subtangent : to the tangent; which is the property of the cycloid, $\frac{1}{2}a$ being $=$ CB, the diameter of the generating circle. Fig. 32.

Now at different distances of O from the node, the cycloid described will be greater in proportion to the sine of AO (fig. 31.); and even in the same cycloid, the latter part will be greater than the former part, as AO grows greater; all the parts of it increasing as the sine of AO increases; and the greatest cycloid will be when A is in the quadratures; and the least when in the syziges, where it is reduced to nothing.

Scholium.

From the foregoing solution, these observations may be made.

1. Tho' the curve R24 has been determined to be a cycloid, yet it is nearer an epicycloid. For at R it sets off nearly in a direction perp to GZ, and during its generation (that is, whilst P performs a semirevolution) the point A moves towards G; and supposing the force at O to be fixed, the last particle of the curve at 4 would be parallel to that at R. But as O really moves forward, some number of degrees, suppose 14, and continues to do so, all the semirevolution; therefore every particle of the curve will have other directions in its description, being more curve than before;

Fig. before; and at laſt the tangents at R and 4, will
32. make an angle of 14 deg. which is the ſame as if an epicycloid was deſcribed on the convex ſide of a circle, going thro' an arch of it equal to 14 degrees.

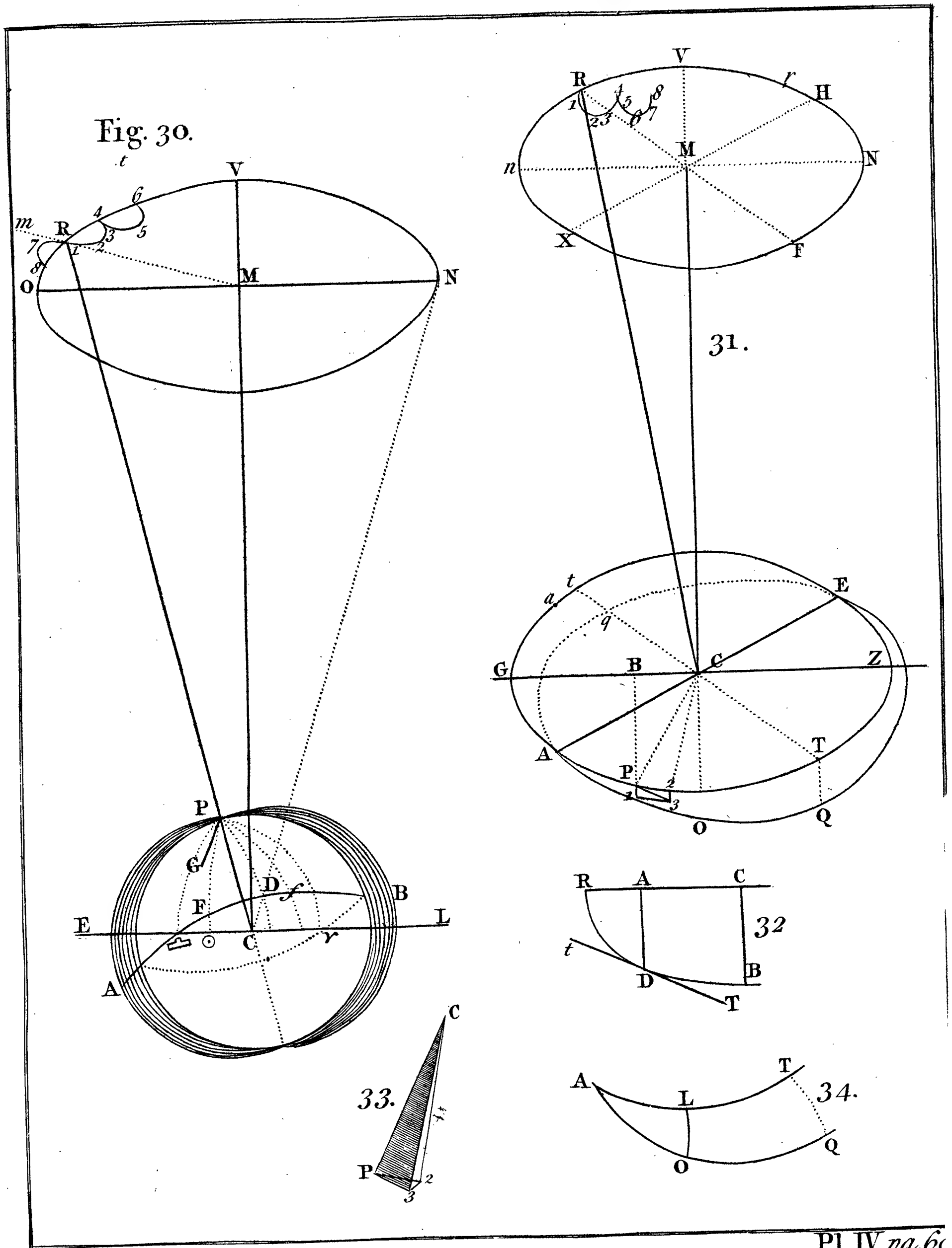
2. Thus the curve deſcribed by R would be nearly an epicycloid, when the force at O is every where of the ſame quantity; yet as O moves about, the force will increaſe and decreaſe in proportion to the ſine of AO; therefore, if you will ſuppoſe ſuch an epicycloid deſcribed as above-mentioned, and moreover imagine the radius of the generating circle to ſwell or increaſe, in the ſame ratio as the S.AO increaſes; then ſuch an epicycloid will nearly repreſent the curve deſcribed by R. For then every part of it will be greater or leſs, in proportion to the force that generates it. But enough of this. All that I ſhall add on this head is the ſolution of the two following problems, upon account of their curioſity, as depending on the foregoing principles.

PROP. XXXIII. *Prob.*

To find the diſturbing force of Jupiter or Saturn, upon the earth in its orbit; having that of the ſun upon the moon given.

26. Let the matter in the ſun and Jupiter be as m to 1. E, I, L the periodic times of the earth, Jupiter and the moon. A, B, the diſtances of the earth and Jupiter from the ſun. D the moon's diſtance from the earth. C, c, the centripetal forces of the ſun and Jupiter.

Then (by Prop. XXVII.) the diſturbing force of S upon P, is $\frac{3PK}{ST}C$ or as $\frac{PT}{ST}C$. Therefore if S be the ſun, and P the moon, the diſturbing force is



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is $\frac{D}{A}C$; but if S be Jupiter, P the earth, T the Fig. 26.
sun; then the force is $\frac{A}{B}c$. That is, the sun's disturbing force upon the moon, is to Jupiter's disturbing force upon the earth; as $\frac{D}{A}C$ to $\frac{A}{B}c$; or as DBC to A^2c. But (Cor. 2. VII.) $C = \frac{m}{AA}$, and $c = \frac{1}{BB}$. Therefore the sun's force upon the moon, is to Jupiter's upon the earth; as $\frac{DBm}{AA}$ to $\frac{AA}{BB}$, or as DB^3m to A^4; that is (Prop. XVII.), as $DI^2m : AE^2$. That is, the sun's disturbing force upon the moon, is to Jupiter's disturbing force upon the earth; as $D \times II \times m$, to $A \times EE$. But that of the moon is known, and consequently that of Jupiter. And if for I and m, we put Saturn's periodic time, and quantity of matter; Saturn's disturbing force will be known.

Cor. 1. *The angular errors generated in the moon by the sun, are to the errors generated in the earth by Jupiter in the same time,* : : *as* $IIL \times m$, *to* E^3.

For (Prop. XXVIII. Cor. 2.) these errors are as the forces and periodic times, divided by the distances. Therefore the sun's effect to Jupiter's, is as $\frac{D \times II \times m \times L}{D}$ to $\frac{A \times E^2 \times E}{A}$; or as $IILm$ to E^3.

Cor. 2. *Hence the error generated in the moon by the sun, is to the error generated in the earth by Jupiter; as* 11230 *to* 1, *and to that generated by Saturn, as* 196076 *to* 1.

For put $I = 4332\frac{1}{2}$ days, $L = 27\frac{1}{3}$, $m = 1067$, $E = 365\frac{1}{4}$; then $\frac{I^2Lm}{E^3} = 11230$, And putting

 I =

Fig. 26. $I = 10759\frac{1}{4}$, and $m = 3021$, for Saturn; then $\frac{I^2Lm}{E^3} = 196076$.

Cor. 3. *The force of Saturn to the force of Jupiter to disturb the earth, is as* 1 *to* $17\frac{1}{2}$.

Cor. 4. *The motion of the nodes of the earth's orbit by Jupiter's action, in* 100 *years, is* $10' 20'' \frac{1}{2}$. *And by Saturn's,* $35'' \frac{2}{3}$.

For the motion of the moon's nodes in a year is $19° 20' 32''$, or $69632''$; this divided by 11230 gives $6''.2005$, multiplied by 100, is $620''.05$, which increased in the ratio of the cosine of inclination of Jupiter's orbit ($1° 19' 10''$), to that of the moon's ($5° 8' \frac{1}{2}$), produces $10' 22'' \frac{1}{2}$. Which diminished in the ratio of 1 to $17\frac{1}{2}$, gives $35'' \frac{2}{3}$ for Saturn.

Cor. 5. *The motion of the earth's aphelion by the action of Jupiter, is* $21' 44''$ *in* 100 *years, in consequentia. And by Saturn,* $1' 14'' \frac{1}{2}$.

For the motion of the moon's apogee is $40° 40' 43''$, or $146443''$ in a year. This divided by 11230 gives $13.04''$; which multiplied by 100, gives $1304''$ or $21' 44''$. And divided by $17\frac{1}{2}$, gives $74'' \frac{1}{2}$.

PROP. XXXIV. *Prob.*

35. *To find the variation of inclination of the earth's orbit, by the action of Jupiter in* 100 *years; and the like for Saturn.*

Let ♈ 69 ♎ ♑ be the ecliptic, or plane of the earth's orbit; GFH the orbit of Jupiter; G Jupiter's ascending node; E, I, Q, the poles of the ecliptic, Jupiter's orbit, and the equator, respectively; ECK a circle parallel to GF; and D*m*Q a circle parallel to the ecliptic. The pole Q here moves regularly along the circle Q*l*D, by the precession

cession of the equinoxes; which circle is no way affected or altered by Jupiter's action; because Jupiter cannot be supposed to have any force to move the equinoctial points, or alter their regular motion. But he has a power of acting upon the whole body of the earth, and altering its orbit, and consequently the pole E of the ecliptic; which pole is therefore made to move along the circle ECK. Therefore we must suppose the orbit of Jupiter fixed, and consequently the pole I, and circle ECK. And now we have to compute the motion of E along the circle ECK. Fig. 35.

The precession of the equinoxes in 100 years is — —	1° 23′ 20″
And (by the last prob.) the motion of Jupiter's nodes in 100 years is	10′ 22″ $\frac{1}{2}$ or 622″ $\frac{1}{2}$
Jupiter's ascending node G (1755, angle QEG) — —	♋ 8° 20′
Inclination of Jupiter's orbit —	1° 19′ 10″.

Therefore make the angle QEa = 1° 23′ $\frac{1}{3}$, and EIC = 10′ 22 $\frac{1}{2}$. Upon Ea let fall the perp. Co; then Eo is the decrease of EQ or Ea, which (Prop. XXXII.) is the same as the decrease of the inclination of the planes of the ecliptic and equinoctial.

In the triangle EIC, by reason of the very small angle EIC, we shall have as rad : S.IC (1° 19′ 10″) : : angle EIC (622″ $\frac{1}{2}$) : EC = $\frac{\text{EIC} \times \text{S.IC}}{\text{rad}}$ = 14″.3. To the angle GEQ (8° 20′), add QEa (1° 23′ $\frac{1}{3}$), then aEG or oEC = 9° 43′ $\frac{1}{3}$. Then in the very small right lined triangle ECo, rad : EC : : cos. oEC (9° 43′ $\frac{1}{3}$) : Eo = $\frac{\text{EC} \times \text{cos.}\, o\text{EC}}{\text{rad}}$ = 14″.1.

Or Eo = $\frac{\text{EIC} \times \text{S.IC} \times \text{cos.}\, o\text{EC}}{\text{rad}^2}$, the decrease of inclination of the equinoctial in 100 years by the

Fig. 35. action of Jupiter. And this decreafe will amount to a minute in 425 years.

If the fame computation was applied to Saturn, putting EIC $= 35'' \frac{2}{3}$, IC $= 2^\circ 30' 10''$, oEC $=$ $(21^\circ 21' 36'' + 1^\circ 23' \frac{1}{3})$ $22^\circ 44' 56''$; the decreafe by Saturn will be $1''.44$. Therefore the decreafe by both will be $15''.54$; which will be a minute in 386 years.

Cor. 1. *The inclination will decreafe till* E *and* a *be at their neareft diftance in the two circles, which will be above* 6000 *years; and then it will increafe again. It has likewife been decreafing for above* 8000 *years.*

For the diameters of the circles EK and DQ, being about as 1 to 17. And the angle IEQ being $81^\circ 40'$, and the difference of the motions of E and Q being $1^\circ 13'$; it will decreafe nearly as many centuries as is the quotient of $81^\circ 40'$ divided by $1^\circ 13'$, which is 67. Alfo the fupplement $98^\circ 20'$ divided by $1^\circ 13'$, gives 81 centuries, it has been decreafing.

Cor. 2. *But the increafe or decreafe for every century is not* $15''.54$, *as determined in this particular fituation. For as it approaches to its maximum or minimum, it varies very flow, and at thefe places is at a ftand for a long time.*

Cor. 3. *The inclination can never be lefs than* $20^\circ 50' 54''$; *nor greater than* $26^\circ 7' 20''$.

For the neareft and greateft diftances of the two circles EK, DQ amount but to thefe. And there muft be many revolutions, before they can light upon thefe two points, if the world can be fuppofed to exift fo long.

SCHOLIUM.

The difturbing forces of Jupiter and Saturn here made ufe of are derived from that of the fun upon on the moon; but these forces are really more than are here determined. For in calculating the disturbing force of the sun (by Prop. XXVII.), the forces upon T and p are as rr to $rr + 2ry + yy$; but by reason of the great distance of the sun, the part yy is left out, as being extremely small. But in the case of Jupiter and Saturn it is otherwise, and therefore yy must be taken in. Whence the disturbing force must have an additional increase, which is as $2ry$ to yy, or as $2r$ to y, which in Jupiter is as 10 to 1, and in Saturn as 19 to 1. Therefore Jupiter's disturbing force must be increased by $\frac{1}{10}$th, and Saturn's by $\frac{1}{19}$th; which being done, their effects will be proportional, and the decrease of inclination of the ecliptic in 100 years, by Jupiter and Saturn will become 15″.45, and 1″.51; and by both 16″.96 or 17″ nearly; which will mount to a minute in 353 years. Fig. 35.

But if the observations of the antients can be depended on, the obliquity decreases faster than this. For by the observations of *Aristarchus*, *Eratosthenes*, *Hyparchus*, *Ptolemy*, and *Theon*, the obliquity was found to be 23° 51′. And none of them lived 300 years before Christ, and two of them after. So that in little more than 2000 years, there is a difference of 22′; which is more than a minute in 100 years, and is more than three times as fast as we have here determined it.

I shall now proceed to some things of another kind, relating to centripetal forces; which as far as I can find, have not been meddled with by any body before.

PROP. XXXV. *Prob.*

If the circle GDFE *be moved along the right line* AB, *whilst it turns round its axis; to find its motion upon a horizontal plane.* 36.

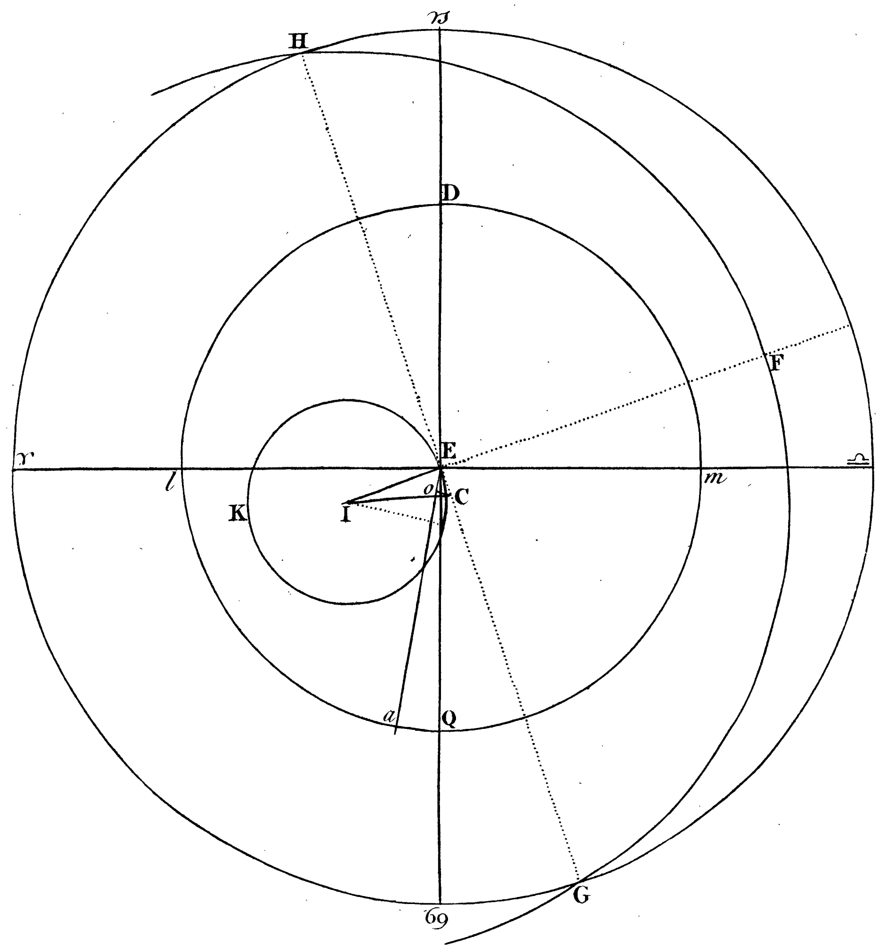
Suppose the circle inlined in any given angle to the horizon, the line of direction AB being at first in

Fig. 36. in its plane; and let it move round its axis CP, which is perpendicular to its plane, with any velocity. Let O be the center of oscillation.

Now the circle will endeavour to descend by its gravity, in the same manner, as the single point O would do. Therefore suppose the point F to descend thro' Ff in a moment of time, and that F is transferred to I in the same time. Then if the parallelogram FfnI be compleated, the point F by the compound motion, will move along Fn. By this means CP, the axis of revolution, will be transferred to the position Cq, inclining more towards B. But when the circle has made half a revolution, and G is come to the place F; the points in F, proceeding in the tract Fn, will move the axis Cq forwards, as before; that is (by the turning of the circle) it will throw it into its former place CP. So that during a revolution, the axis is thrown contrary ways in all the opposite points, and so is always restored back to its first place in regard to the plane. Therefore the circle always revolves about the axis CP, whilst CP continually inclines more and more forward; that is, the plane of the circle continually alters its position; and the variation of its position is known from the lines FI, nI; and is equal to the angle nFI or PCq. And since the circle endeavours to move along a line which is in the plane nFG, it will no longer go along AB, but deviate from it into a new tract, which is now to be found.

It may be convenient to imagine the circle to be a poligon with an infinite number of sides. Then let KL be the horizon, Mg the plane of the the circle, g being a point in the right line AB; let the next point of the circle (or angle of the poligon) descend to the horizon, thro' the very small space rt; then in the right angled triangle trg, S. inclination (tgr) : tr or nI (which is as S. nFI) : : rad

Fig. 35.



Cent. Forces

rad (1) : $tg = \frac{S.nFI}{S.tgr}$. Therefore if the circle be moved thro' Gg in the same time, then from g (in the line AB) setting off $gt = \frac{S\ nFI}{S.\text{inclination}}$, then t will be a point in the curve GtR, thro' which the circle will pass. After the same manner it will again deviate from the last direction tb; and describe the curve GRVWX. Fig. 36.

Generally the whirling motion round its axis, is equal to its progressive motion; for the friction of the plane soon reduces it to that. But take away the friction, and these two different motions may be what you will.

Cor. 1. *The curvature in any place, is reciprocally as these three quantities, the velocity of rotation, the progressive velocity, and the tangent of inclination.*

For the curvature is as the angle tGg, that is, as $\frac{tg}{Gg}$ or as $\frac{S.nFI}{Gg \times S.\text{inclination}}$; that is, as $\frac{nI}{FI \times Gg \times S.\text{inclin.}}$; that is, as $\frac{\text{Cos. incl.}}{FI \times Gg \times S.\text{inclin.}}$ or as $\frac{1}{FI \times Gg \times \tan.\ \text{inclin.}}$. For nI is as the cosine of inclination, being the space described upon the inclined plane nI.

Cor. 2. *Taking away all impediments, the circle always keeps the same inclination to the horizon.*

For the position of the plane FnI is such, that the axes CP, Cq, are both parallel to it. If we suppose gravity to act by a single impulse at O, then F will move to n, and P to q. And the plane of the circle endeavouring to descend a little at D, and rise at E; a new point of the circle as t, lying beyond G will instantly touch the plane; by which means it leaves the line AG. And since at every

Fig. 36. every point of contact as G, the pole P moves (at each impulse of gravity) in a line perp. to CG, and also parallel to the tangent arch at G; and the like at every new point of contact; it is plain CP is always alike inclined to the horizon. Consequently, when gravity is continual, the circle coming continually to new points of contact, the axis CP will always revolve round at the same inclination, and therefore the plane of the circle will also have the same inclination to the horizon.

This might also be proved after the manner of the XXXth Prop. not considering the progressive motion of the circle.

All this is supposing there is no resistance, friction, or other irregularity. But since in fact, the resistance of the air continually lessens its motion, and the smoothness of the plane it runs on, causes the foot or bottom of the circle to slide outward, which continually lessens the inclination, and brings the axis more upright; and the more oblique the plane of the circle is, the faster it slides out. Upon these accounts it can never describe a circle, but only a sort of spiral line; and the plane of the circle descending lower and lower, at last falls flat upon the horizon.

Cor. 3. *Hence a circle moving without any resistance, &c. upon a horizontal plane; will describe a circle upon that plane.*

For the velocity and inclination continuing the same; the curvature of the tract described, will be every where alike.

Cor. 4. *And to find the diameter of the circle or orbit described.*

Let t be a very small part of time wherein Gg is described, v the velocity of projection per second, b the space descended by gravity in time t, s and c the sine and cosine of the circles inclination, $f =$

16

$16\frac{1}{12}$ feet; then will cb = ſpace deſcended along Fig.
the inclined plane Ff; and by the laws of motion, 36.

$$1'' : v :: t'' : Gg = tv.$$

$$\text{and } tt : b :: 1'' : f = \frac{b}{tt}.$$

Then whilſt O has moved thro' the length Gg, t or I has deſcended thro' the ſpace cb on an inclined plane parallel to Ff. But we proved $gt = \frac{nI}{s} = \frac{tr}{s} = \frac{cb}{s}$. And therefore the diameter of the orbit $= \frac{\overline{Gg}^2}{tg} = \frac{\overline{Gg}^2 \times s}{cb} = \frac{ttvvs}{cb} = \frac{vvs}{cf} = \frac{vv}{f} \times$ tan. inclination.

Cor. 5. *Alſo to find the periodic time, or time of one revolution.*

Let D = diameter of the orbit, then by Cor. laſt, $\frac{\overline{Gg}^2 \times s}{cb} = D$, and $Gg = \sqrt{\frac{cbD}{s}}$. The circumference of the orbit is $\frac{\pi \times \overline{Gg}^2 \times s}{cb}$ (putting $\pi = 3.1416$); and by uniform motion, $Gg : t :: \frac{\pi \times \overline{Gg}^2 \times s}{cb} : \frac{\pi ts \times Gg}{cb}$ the periodic time $= \frac{\pi ts}{cb} \sqrt{\frac{cbD}{s}} = \pi \sqrt{\frac{ttsD}{bc}} = \pi \sqrt{\frac{SD}{fc}}$.

Or thus,

$Gg\ (tv) : t ::$ circumference $\frac{\pi vvs}{fc}$: periodic time $= \frac{\pi vs}{fc} = \frac{\pi v}{f} \times$ tan. inclination.

PROP.

Fig.

PROP. XXXVI. *Prob.*

37. *If* DEF *be the ſurface of a right cone, whoſe axis* AE *is perpendicular to the horizon, and* DHFG *a circular plane parallel to the horizon; and if a circle* ab *revolves round in the periphery* DHFG, *with its axis* tBe *always parallel to the ſide of the cone* DE, *where it then is. To find the periodic time of* ab, *in the circle* DHFG.

Draw DBA perp. to DE, and BC perp. to AC. Let $\pi = 3.1416$, $h =$ ſpace deſcended by gravity in the time t.

Then if the force of gravity be repreſented by AC, the centrifugal force at B to keep the circle *ab* in that poſition, thro' its whole revolution, will be denoted by BC. Then it will be AC (the gravity) : BC (the cent. force) : : $h : \frac{BC}{AC}h =$ ſpace deſcended towards C, by the force in direction BC = the effect of the centrifugal force. Therefore by (Prop. II.) the periodic time is $\pi t\sqrt{\dfrac{2BC}{\frac{BC}{AC}\times h}} = \pi t\sqrt{\dfrac{2AC}{h}}$.

Cor. 1. *Hence the periodic time* $= \pi t\sqrt{\dfrac{2BC}{h}\times \textit{tan. inclination } ABC}$.

For $S.A : S.B :: BC : AC = \dfrac{S.B}{S.A}\times BC = \dfrac{S.B}{\text{coſ.}B}\times BC = \text{tan. } B\times BC$; whence, the periodic time $= \pi t\sqrt{\dfrac{2BC}{h}\times \text{tan. } ABC}$.

Cor. 2. *Draw* BI *parallel to* DE, *and* DL *parallel to* BC; *then the periodic time* $= \dfrac{\pi t\times BC\times\sqrt{2}}{\sqrt{CI}\times h}$.

For $AC = \dfrac{BC^2}{CI}$.

Cor.

Cor. 3. *Hence the curve described on the conic surface, is the same as that described on a horizontal plane, as explained in the last Prop. All the difference is, that the conic surface hinders the circle ab from sliding outward, which the horizontal plane cannot do; except it be supposed to be so rough, that the circle cannot slide on it.* Fig. 37.

For in Cor. 4. of the last Prop. v is put for the velocity of G along AB; but putting it for the velocity at C, you'll have the diameter of the orbit passing thro' C: that is, (fig. 37.) instead of 2DL we should find 2BC for the diameter D of the orbit. 36. 37.

And by Cor. 5, the periodic time is $\pi t\sqrt{\frac{sD}{bc}}$. And in Cor. 1. of this Prop. the periodic time is $\pi t \sqrt{\frac{2BC}{b}} \times$ tan. inclination; which is equal to the former, because $D = 2BC$, and $\frac{s}{c} =$ tan. of the inclination. So the curve is the same in both cases.

PROP. XXXVII. *Prob.*

To explain the motion of a top, or such like whirling body. 38.

Let ABC be a top whirling about in the order AEC; FDG a circle described by any point D in the surface, K its center of gravity, BKS the axis of the top.

If the top be upright upon the foot B; that is, if BS be perp. to the horizon, and moves swiftly about; it will continue upright till the motion slacken. But when it is going to fall, it will lean to one side; therefore suppose D to be the lowest point in the circle FG; then the top endeavours by its gravity to descend towards D. Let the force of gravity alone move the point D thro' the space Do, in a very small time; during which, the rotary motion

Fig. 38. motion would carry the point D to *r*. Compleat the parallelogram D*rpo*, and the point D will be carried thro' D*p*; that is, the circle FDG will come into the poſition D*p*; and therefore the axis BKS (perp. to the circle FG) will be moved in a direction towards H, perp. to DK; and the point S moved to *n*; the particle S*n* being parallel to D*p*. After the ſame manner by a new impulſe of gravity at *p*, the loweſt point; the circle FDG, will be moved into a new poſition, below D*p*, and the point S carried from *n* to *t*. And ſo by every impulſe of gravity, the point S will be moved gradually forward, thro' the circle S*ntqlz*; and thus the top recovers itſelf from falling; the motion of S being always parallel to that of D. And therefore the motion of the axis BS will be the ſame way about, as the top's motion is. And thus the point S will continue to make ſeveral revolutions by a ſlow motion, whilſt the top makes its revolutions about its axis, by a ſwift motion.

Cor. 1. *This motion of the top and its axis, is ſimilar to the motion of an oblong ſpheroid, and its nodes.*

For (Cor. 6. Prop. XXX.) the nodes move the ſame way about as the body revolves, and ſo does the axis of the top; and therefore this motion may be called the motion of the nodes of the top.

Cor. 2. *When the tops motion is very ſwift, the circle* S*ql* *is very ſmall; and as it grows ſlower, that circle will grow bigger and bigger, till the top falls.*

For when the top's motion is very ſwift, D*r* will be greater, and the angle *r*D*p* leſs; and the circle D*p* will deviate leſs from DG. And gravity having little power to diſturb its motion, the circle S*ql* will be extremely ſmall, and the top will revolve about the axis in appearance unmoved. But as the top's

top's motion by resistance and friction grows less Fig.
and less, D*r* will be less; and the circle D*p* will de- 38.
viate more from DG; that is, gravity will have more and more power to disturb its motion; and the axis BS will describe a greater and greater circle with the point S, or rather a spiral, till at last the top falls down.

Cor. 3. *As the top grows slow, and the motion weak, and the pole* S *describes greater and greater circles, the foot* B *is thrown out to the opposite side, describing a circle* B*b*, *which is greater as* S*ql* *is greater; and goes the same way about.* 39.

For the center of gravity K always endeavours to be at rest, whilst the body revolves about. Therefore when the top grows weak, and the pole S describes greater circles, the foot B is thrown further out to the opposite side; and being always opposite, will describe a circle proportional to S*ql*; the foot B going the same way about as S does. And these circles B*b* will continually grow greater and greater till the top falls down. Till then the top rolls about and about from the position CAB to the opposite position *cab*, till the motion end, and the top falls down. And these are the principal phœnomena of the motion of a top.

F I N I S.

ERRATA.

Page	Line	read
1	11	(Preface) irregularities only
59	19	P♎C and P♈C.
65	1	4, R recedes from M,
78	6	its axis 1 B2

In the Plates.

Fig. 5. *m* ſhould be ſhaded.
Fig. 23. P*n* ſhould be perp. to F*p*.

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